KAIST POW - Fall 2025

2024**** 정영훈

woadwot2718@kaist.ac.kr

2025-14

Show that any set of $d+2$ points in $\mathbb{R}^d$ can be partitioned into two sets whose convex hulls intersect.

Solution

This is known as Radon's theorem. Let $x_1, \ldots, x_{d+2} \in \mathbb{R}^d$ be $d+2$ points in $\mathbb{R}^d$. Consider the $d+2$ vectors $(x_i, 1) \in \mathbb{R}^{d+1}$. Since $\mathbb{R}^{d+1}$ has dimension $d+1 < d+2$, these vectors are linearly dependent, so there exist scalars $\lambda_1, \ldots, \lambda_{d+2}$, not all zero, such that

$$\sum_{i=1}^{d+2} \lambda_i (x_i, 1) = 0.$$

That is, $\sum_{i=1}^{d+2} \lambda_i x_i = 0$ and $\sum_{i=1}^{d+2} \lambda_i = 0$.

Let $I_+ := \{i : \lambda_i > 0\}$ and $I_- := \{i : \lambda_i < 0\}$. The equation $\sum_i \lambda_i = 0$ and nonzeroness implies both I_+ and I_- are nonempty. Let

$$S_+ := \sum_{i \in I_+} \lambda_i = \sum_{i \in I_-} (-\lambda_i) =: S_- > 0$$

Also, define $\alpha_i := \frac{\lambda_i}{S_+}$ $(i \in I_+)$, $\beta_i := \frac{-\lambda_i}{S_-}$ $(i \in I_-)$. Then $\alpha_i, \beta_i \geq 0$ and $\sum_{i \in I_+} \alpha_i = \sum_{i \in I_-} \beta_i = 1$. Using $\sum_i \lambda_i x_i = 0$, we get

$$\sum_{i \in I_+} \alpha_i x_i = \frac{1}{S_+} \sum_{i \in I_+} \lambda_i x_i = \frac{1}{S_+} \sum_{i \in I_-} (-\lambda_i) x_i = \sum_{i \in I_-} \beta_i x_i =: y.$$

Thus the point y is a convex combination of $\{x_i : i \in I_+\}$ and also a convex combination of $\{x_i : i \in I_-\}$. Therefore

$$y \in \operatorname{conv}\{x_i : i \in I_+\} \cap \operatorname{conv}\{x_i : i \in I_-\},$$

so the convex hulls of the two parts intersect. (Any x_i with $\lambda_i = 0$ can be placed in either part without affecting y.)

Note. The problem doesn't hold for $d+1$ points in $\mathbb{R}^d$

Let $v_0 = 0$, $v_1 = e_1, \ldots, v_d = e_d \in \mathbb{R}^d$. Let A, B be a partition of $\{v_0, \ldots, v_d\}$ into two nonempty sets. Suppose, for contradiction, that $\text{conv}(A) \cap \text{conv}(B) \neq \varnothing$. Then there exists y such that

$$y = \sum_{i \in A} \alpha_i v_i = \sum_{j \in B} \beta_j v_j \quad \text{and} \quad \sum_{i \in A} \alpha_i = \sum_{j \in B} \beta_j = 1.$$

where $\alpha_i, \beta_j \geq 0$ By adding auxiliary zeros, we obtain $(d+1)$-tuples $(\alpha_0, \ldots, \alpha_d)$ and $(\beta_0, \ldots, \beta_d)$ with $y = \sum_{i=0}^{d} \alpha_i v_i = \sum_{i=0}^{d} \beta_i v_i$ and $\sum_i \alpha_i = \sum_i \beta_i = 1$. i.e.

$$\sum_{i=1}^{d} (\alpha_i - \beta_i) v_i = 0 \quad \text{and} \quad \sum_{i=0}^{d} (\alpha_i - \beta_i) = 0.$$

By the independence of standard basis, first equation implies that $\alpha_i - \beta_i = 0$ for all $1 \leq i \leq d$ and the second equation additionaly shows that $\alpha_0 - \beta_0$ is also zero.

Note that $A \cap B = \varnothing$ says that at least one of α_i or β_i must be zero for all i. Combining this observation with the equation $\alpha_i = \beta_i$, we get $\alpha_i = \beta_i = 0$ for all k, contradicting to the fact that the sums equal 1. Therefore $\text{conv}(A) \cap \text{conv}(B) = \varnothing$ for every nontrivial partition.