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Let Z be a standard normal random variable. The moment generating function of Z is $\mathbb{E}[e^{tZ}] = e^{t^2/2}$, so by Markov's inequality, for any $t, u > 0$ it holds that

$$\mathbb{P}(Z > u) = \mathbb{P}(e^{tZ} > e^{tu}) \le \frac{\mathbb{E}[e^{tZ}]}{e^{tu}} = e^{-tu+t^2/2}.$$

In particular, by considering when $t = u$, we get

$$\mathbb{P}(Z > t) \le e^{-t^2/2}. \tag{1}$$

Meanwhile, for any nonnegative random variable Y, we have

$$\mathbb{E}[Y] = \mathbb{E}\left[\int_0^\infty \mathbb{1}_{\{t<Y\}}(t)\,dt\right] = \int_0^\infty \mathbb{E}\left[\mathbb{1}_{\{t<Y\}}(t)\right]dt = \int_0^\infty \mathbb{P}(Y > t)\,dt.$$

Therefore, we conclude that

$$\begin{aligned}
\mathbb{E}\left[\max_i \frac{|X_i|}{\sqrt{1+\log i}}\right] &= \int_0^\infty \mathbb{P}\left(\max_i \frac{|X_i|}{\sqrt{1+\log i}} > t\right)dt \\
&\le \int_0^2 1\,dt + \int_2^\infty \mathbb{P}\left(\max_i \frac{|X_i|}{\sqrt{1+\log i}} > t\right)dt \\
&= 2 + \int_2^\infty \mathbb{P}\left(\bigcup_{i=1}^\infty \left\{\frac{|X_i|}{\sqrt{1+\log i}} > t\right\}\right)dt \\
&\le 2 + \int_2^\infty \sum_{i=1}^\infty \mathbb{P}\left(\frac{|X_i|}{\sqrt{1+\log i}} > t\right)dt \\
&= 2 + \int_2^\infty \sum_{i=1}^\infty 2\,\mathbb{P}\left(X_i > t\sqrt{1+\log i}\right)dt \\
&\le 2 + 2\int_2^\infty \sum_{i=1}^\infty e^{-t^2(1+\log i)/2}\,dt \\
&= 2 + 2\int_2^\infty \sum_{i=1}^\infty (ei)^{-t^2/2}\,dt \\
&\le 2 + 2\int_2^\infty \sum_{i=1}^\infty e^{-t^2/2} i^{-2}\,dt \\
&= 2 + \frac{\pi^2}{3}\int_2^\infty e^{-t^2/2}\,dt \\
&\le 2 + \frac{\pi^{5/2}}{3\sqrt{2}}
\end{aligned}$$

where in the third inequality we used (1), in the fourth the fact that $t^2/2 \ge 2$ as the domain of the integral is $t \ge 2$, and in the fifth the Gaussian integral $\int_0^\infty e^{-t^2/2}\,dt = \sqrt{\pi/2}$.