# POW 2025-10

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## 1 Problem Statement

Let $P$ be a regular $2n$-gon. A *perfect matching* consists of $n$ chords connecting the $2n$ vertices in pairs. Let $X$ be the random number of intersection points formed by these chords when the perfect matching is chosen uniformly at random. We seek to determine

$$\lim_{n\to\infty} \frac{\mathbb{E}[X]}{n^2}.$$

**Remark 1.** If more than two chords intersect at the same point, this intersection point is counted only once. We denote by $X_{\text{true}}$ the number of distinct geometric intersection points.

## 2 Analysis of Pairwise Intersections

We begin by calculating the expected number of *pairs* of intersecting chords, denoted $X_{\text{pairs}}$.

**Lemma 1.** *Consider any two chords in a given perfect matching connecting four distinct vertices. The probability that these two chords intersect is* $\frac{1}{3}$.

*Proof.* Let the four vertices be $v_1, v_2, v_3, v_4$ in clockwise order around the polygon. There are three ways to form two chords using these four vertices:

(1) $(v_1, v_2)$ and $(v_3, v_4)$: These chords do not intersect.

(2) $(v_1, v_4)$ and $(v_2, v_3)$: These chords do not intersect.

(3) $(v_1, v_3)$ and $(v_2, v_4)$: These chords intersect.

Since the perfect matching is chosen uniformly at random, each of these three patterns is equally likely for any specific set of 4 vertices that are endpoints of two chords in the matching. ∎

**Proposition 1.** *The expected number of pairs of intersecting chords is*

$$\mathbb{E}[X_{pairs}] = \frac{n(n-1)}{6}.$$

*Proof.* A perfect matching has $n$ chords, giving $\binom{n}{2}$ pairs of chords. Let $I_{ij}$ be an indicator random variable that equals 1 if chords $i$ and $j$ intersect, and 0 otherwise. Then

$$X_{\text{pairs}} = \sum_{1\le i<j\le n} I_{ij}.$$

By linearity of expectation and Lemma 1:

$$\mathbb{E}[X_{\text{pairs}}] = \sum_{1\le i<j\le n} \mathbb{E}[I_{ij}] = \binom{n}{2}\cdot\frac{1}{3} = \frac{n(n-1)}{6}.$$

∎

## 3 Alternative Calculation via Vertex Quadruplets

We can verify this result by considering quadruplets of vertices directly.

**Lemma 2.** *The probability that a specific quadruplet of vertices forms an intersection in a random perfect matching is*

$$\frac{(2n-5)!!}{(2n-1)!!} = \frac{1}{(2n-1)(2n-3)}.$$

*Proof.* Any intersection point is defined by 4 vertices $v_1, v_2, v_3, v_4$ in cyclic order, where the intersecting chords are $(v_1, v_3)$ and $(v_2, v_4)$.

The total number of perfect matchings on $2k$ vertices is $M_{2k} = (2k-1)!! = (2k-1)(2k-3)\cdots 1$.

For the specific quadruplet to form an intersection, we need chords $(v_1, v_3)$ and $(v_2, v_4)$ to be present, with the remaining $2n-4$ vertices forming a perfect matching among themselves. The number of such matchings is $M_{2n-4} = (2n-5)!!$.

Therefore, the probability is $\frac{M_{2n-4}}{M_{2n}} = \frac{(2n-5)!!}{(2n-1)!!}$. ■

**Proposition 2.** *Using the quadruplet approach:*

$$\mathbb{E}[X_{pairs}] = \binom{2n}{4} \cdot \frac{1}{(2n-1)(2n-3)} = \frac{n(n-1)}{6}.$$

*Proof.*

$$\mathbb{E}[X_{\text{pairs}}] = \binom{2n}{4} \cdot \frac{1}{(2n-1)(2n-3)} \tag{1}$$

$$= \frac{2n(2n-1)(2n-2)(2n-3)}{24} \cdot \frac{1}{(2n-1)(2n-3)} \tag{2}$$

$$= \frac{2n(2n-2)}{24} = \frac{4n(n-1)}{24} = \frac{n(n-1)}{6}. \tag{3}$$

■

## 4 From Pairwise Intersections to Distinct Points

Now we address the relationship between $X_{\text{pairs}}$ and $X_{\text{true}}$.

If $k_P$ chords intersect at a single geometric point $P$, these chords form $\binom{k_P}{2}$ pairwise intersections. Such a point $P$ contributes 1 to $X_{\text{true}}$ but $\binom{k_P}{2}$ to $X_{\text{pairs}}$.

We have:

$$X_{\text{pairs}} = \sum_{P \text{ distinct intersection point}} \binom{k_P}{2} \tag{4}$$

$$X_{\text{true}} = \sum_{\substack{P \text{ distinct intersection point} \\ k_P \geq 2}} 1 \tag{5}$$

Therefore:

$$\mathbb{E}[X_{\text{pairs}}] - \mathbb{E}[X_{\text{true}}] = \mathbb{E}\left[\sum_P \left(\binom{k_P}{2} - 1\right)\right]$$

This difference is non-zero only for points $P$ where $k_P \geq 3$.

**Lemma 3.** *For a regular $2n$-gon, the expected contribution from points where three or more chords intersect is $O(1)$.*

*Proof.* The primary difference between $\mathbb{E}[X_{\text{pairs}}]$ and $\mathbb{E}[X_{\text{true}}]$ arises from multiple chords intersecting at a single point. For a regular $2n$-gon, any intersection of three or more chords typically occurs at the center of the polygon, formed by diameters.

Let $N_D$ be the number of diameters in the random matching. An intersection point $P$ contributes 1 to $X_{\text{true}}$ (if $k_P \geq 2$) and $\binom{k_P}{2}$ to $X_{\text{pairs}}$. The sum $\sum_P \left(\binom{k_P}{2} - 1\right)$ accounts for the difference.

For points $P$ not at the center, $k_P \leq 2$ (assuming general position for non-diametral chords), so $\binom{k_P}{2} - 1 = 0$. Thus, the main correction term comes from the center $O$:

$$\left(\binom{N_D}{2} - X_c\right),$$

where $X_c = 1$ if $N_D \geq 2$ and 0 otherwise.

We need to show that $\mathbb{E}\left[\binom{N_D}{2}\right]$ and $\mathbb{E}[X_c] = P(N_D \geq 2)$ are $O(1)$.

The number of possible diameters is $n$. Let $d_i$ denote the $i$-th diameter. The probability that $k$ specific diameters are in the matching is

$$\frac{(2n-2k-1)!!}{(2n-1)!!}.$$

The expected number of pairs of distinct diameters $(d_i, d_j)$ in the matching is:

$$\mathbb{E}\left[\sum_{i \neq j} \mathbf{1}_{d_i \in M, d_j \in M}\right] = \binom{n}{2} P(d_1 \in M, d_2 \in M) \tag{6}$$

$$= \binom{n}{2} \frac{(2n-5)!!}{(2n-1)!!} \tag{7}$$

$$= \frac{n(n-1)}{2} \frac{1}{(2n-1)(2n-3)}. \tag{8}$$

This quantity is $\mathbb{E}\left[\binom{N_D}{2}\right]$. As $n \to \infty$:

$$\mathbb{E}\left[\binom{N_D}{2}\right] \sim \frac{n^2/2}{4n^2} = \frac{1}{8},$$

which is $O(1)$. Since $P(N_D \geq 2) \leq 1$, it is also $O(1)$.

More precisely, $N_D$ asymptotically follows a Poisson distribution with mean

$$\lambda = \lim_{n\to\infty} n \cdot P(d_1 \in M) \tag{9}$$

$$= \lim_{n\to\infty} n \frac{(2n-3)!!}{(2n-1)!!} \tag{10}$$

$$= \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2}. \tag{11}$$

Thus:

$$\lim_{n\to\infty} \mathbb{E}\left[\binom{N_D}{2}\right] = \mathbb{E}\left[\binom{Z}{2}\right]$$

where $Z \sim \text{Poisson}(1/2)$, which equals $\frac{\lambda^2}{2} = \frac{(1/2)^2}{2} = \frac{1}{8}$.

And:

$$\lim_{n\to\infty} P(N_D \geq 2) = P(Z \geq 2) \tag{12}$$

$$= 1 - e^{-1/2} - \frac{1}{2}e^{-1/2} \tag{13}$$

$$= 1 - \frac{3}{2}e^{-1/2}. \tag{14}$$

Both terms are $O(1)$. Therefore, $\mathbb{E}[X_{\text{pairs}}] - \mathbb{E}[X_{\text{true}}] = O(1)$. ■

## 5 Main Result

**Theorem.**

$$\lim_{n\to\infty} \frac{\mathbb{E}[X]}{n^2} = \frac{1}{6}.$$

*Proof.* From our analysis:

$$\mathbb{E}[X_{\text{true}}] = \mathbb{E}[X_{\text{pairs}}] - O(1) = \frac{n(n-1)}{6} - O(1) = \frac{n^2}{6} - \frac{n}{6} - O(1).$$

Therefore:

$$\lim_{n\to\infty} \frac{\mathbb{E}[X]}{n^2} = \lim_{n\to\infty} \frac{\frac{n^2}{6} - \frac{n}{6} - O(1)}{n^2} \tag{15}$$

$$= \lim_{n\to\infty} \left(\frac{1}{6} - \frac{1}{6n} - O\left(\frac{1}{n^2}\right)\right) \tag{16}$$

$$= \frac{1}{6}. \tag{17}$$

■