POW 2025-09: abc-functions

20230660 정서윤

Problem. For given $a, b \in \boldsymbol{R}$ and $c \in \boldsymbol{Z}$, find all function $f: \boldsymbol{R} \to \boldsymbol{R}$ which is continuous at 0 and satisfies

$$f(ax) = f(bx) + x^c, \qquad \forall\, x \in \boldsymbol{R}\setminus\{0\}$$

Solution.

If $a = b$, then $f(ax) = f(ax) + x^c$, $x^c \equiv 0$, which is contradiction. $\therefore a \neq b$.

If $c \leq 0$, then $\lim_{x\to 0}\big(f(ax) - f(bx)\big) = 0 = \lim_{x\to 0} x^c$, which is contradiction. $\therefore c > 0$.

Case1) $|\boldsymbol{a}| < |\boldsymbol{b}|$

It' easy to show that the $\frac{x^c}{a^c - b^c}$ satisfies all conditions.

Let $f(x)$ be a function satisfying all conditions. Define $g(x) = f(x) - \frac{x^c}{a^c - b^c}$.

Then, we have $g(ax) = g(bx)$ and $g(x)$ is continuous at 0.

If $a = 0$, then $g(bx) = g(0)$, so $g(x) = g(0)$.

Now, we may assume $a \neq 0$. Then, $b \neq 0$ and we have $g(x) = g\left(\frac{a}{b}x\right) = g\left(\frac{a^2}{b^2}x\right) = \cdots = g(\frac{a^n}{b^n}x)$ for all $n \in \boldsymbol{N}$ and for all $x \neq 0$.

Hence, $g(x) = \lim_{n\to\infty} g(\frac{a^n}{b^n}x) = g(0)$ for all $x \neq 0$ since $g(x)$ is continuous at 0.

Therefore, $g(x) = C$ for any constant C. Thus, $f(x) = \frac{x^c}{a^c - b^c} + C$ for any constant C.

Conversely, it satisfies all conditions.

Case2) $|\boldsymbol{a}| > |\boldsymbol{b}|$

Similarly, we can show that $f(x) = \frac{x^c}{a^c - b^c} + C$ for any constant C.

Case3) $|\boldsymbol{a}| = |\boldsymbol{b}|$ **i.e.,** $\boldsymbol{a} = -\boldsymbol{b}$

$f(ax) = f(-ax) + x^c$, so $f(-ax) = f(ax) + (-x)^c$. Hence, $x^c + (-x)^c = 0$, so c is odd. Then, $\frac{x^c}{a^c - b^c}$ is well-defined and satisfies all conditions.

Let $f(x)$ be a function satisfying all conditions. Define $g(x) = f(x) - \frac{x^c}{a^c - b^c}$.

Then, we have $g(ax) = g(-ax)$ for all $x \neq 0$, so $g(x) = g(-x)$. Hence, $g(x)$ is even and continuous at 0. Conversely, $f(x) = \frac{x^c}{a^c - b^c} + g(x)$ satisfies all conditions for any even function $g(x)$ which is continuous at 0.

To sum up, we have $a \neq b$, $c > 0$ and all possible functions are

$$f(x) = \begin{cases} \dfrac{x^c}{a^c - b^c} + C & if\ |\boldsymbol{a}| \neq |\boldsymbol{b}| \\ \dfrac{x^c}{a^c - b^c} + g(x) & if\ |\boldsymbol{a}| = |\boldsymbol{b}|\ and\ c\ is\ odd \end{cases}$$

for any constant C and for any even function $g(x)$ which is continuous at 0.