POW 2025-08: Chordial relations

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First, there should be little change in problem:

$a_1 = 1$, definitely(adding no chords to triangle). For convenience, let $a_0 = 1$.

Suppose $n \geq 2$. Label points by $P_1, \cdots, P_{n+2}$, by clockwise. Divide to two cases,

(1) P_1 is not connected to any other points

Then it is same with adding chords in $(n+1)-$gon with points except P_1.

But in this case, P_2 and P_{n+2} are not connected, so we can either connect two points or not.

Therefore, there are total $2a_{n-1}$ cases in here.

(2) P_1 is connected to some other points

Let $3 \leq k \leq n+1$ be the smallest index that P_k is connected with P_1.

- If $k = 3$, then it is same with adding chords in $(n+1)-$gon with points except P_2.

  So there are a_{n-1} cases in here.

- If $4 \leq k \leq n+1$, then chord $\overline{P_1P_k}$ divides polygon into $k-$gon and $(n+4-k)$-gon.

  (As $\overline{P_1P_k}$ is added, which is counted twice, so sum of sides of two polygons is $n+4$)

  For $(n+4-k)$-gon, there is no restriction, so there are a_{n+2-k} cases.

  For $k-$gon, P_1 should not be connected with $P_2, \cdots, P_{k-1}$.

  Then it is same with (1), so there are $2a_{(k-2)-1} = 2a_{k-3}$ cases.

  Therefore, there are $2a_{k-3}a_{n+2-k}$ total cases.

Adding all cases give $a_n = 2a_{n-1} + a_{n-1} + \sum_{k=4}^{n+1} 2a_{k-3}a_{n+2-k}$.

Putting $\sum_{k=4}^{n+1} 2a_{k-3}a_{n+2-k} = \sum_{k=1}^{n-2} 2a_k a_{n-1-k} = -4a_{n-1} + \sum_{k=0}^{n-1} 2a_k a_{n-1-k}$, then we get $a_n = -a_{n-1} + \sum_{k=0}^{n-1} 2a_k a_{n-1-k}$. Note that this also holds for $n = 1$, recurrence relation of a_n is

$$a_0 = 1, \quad a_{n+1} = -a_n + \sum_{k=0}^{n} 2a_k a_{n-k} \quad (n \geq 0)$$

Now find generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$. As $\{f(x)\}^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n$,

we get the equation $f(x) = -xf(x) + 2x\{f(x)\}^2 + 1$, $2x\{f(x)\}^2 - (x+1)f(x) + 1 = 0$.

Solve for $f(x)$, then we get $f(x) = \dfrac{(x+1) \pm \sqrt{(x+1)^2 - 8x}}{4x} = \dfrac{2}{(x+1) \mp \sqrt{(x+1)^2 - 8x}}$.

Considering $1 = a_0 = \lim_{x \to 0} f(x)$, only $+$ is possible in second form (i.e. only $-$ is possible in first form). Therefore, generating function is

$$f(x) = \frac{(x+1) - \sqrt{x^2 - 6x + 1}}{4x}. \quad \blacksquare$$