POW2025-07 DO COVERS INDUCE INJECTIVE MAPS ON HOMOLOGY

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Problem. Let X and Y be closed manifolds, and suppose X is a cover of Y. Prove or disprove that the induced map on the first homology is injective.

Solution. We disprove it, by the following example.

Denote by Σ_g the orientable surface of genus g (i.e., the connected sum of g distinct tori). Then there is a 2-sheeted cover $\Sigma_3 \to \Sigma_2$, and since $H_1(\Sigma_g) = \mathbb{Z}^{2g}$, the induced map on first homology groups is $\mathbb{Z}^6 \to \mathbb{Z}^4$. Clearly, such map cannot be injective.

Note that we can generalize the example above to an *n*-sheeted cover $\Sigma_{mn+1} \to \Sigma_{m+1}$. This covering map induces the homology map $\mathbb{Z}^{2mn+2} \to \mathbb{Z}^{2m+2}$, which is not injective for n > 1.

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