Kaist POW – Spring 2025

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There are $n+1$ hats, each labeled with a number from 1 to $n+1$, and n people. Each person is randomly assigned exactly one hat, and each hat is assigned to at most one person (i.e., the assignment is injective). A person can see all other assigned hats but cannot see their own hat and the unassigned hat. Each person must independently guess the number on their own hat.

If everyone correctly guesses their own hat's number, they win; otherwise, they lose.
They may discuss a strategy before the hats are assigned, but no communication is allowed afterward.
Determine a strategy that maximizes their probability of winning.

Solution

Denote people by $P_1, \cdots, P_n$. By considering the unassigned hat as P_{n+1}, their given assignment can be realized as an element $\sigma \in S_{n+1}$ and clearly it is 1-1 correspondence. For each person P_i, they have exactly two inconclusive locations, their own hat and the unassigned one. Then this gives two possibilities, say $\alpha_i, \beta_i \in S_{n+1}$. Since α_i and β_i differ by exactly one transposition, one may assume that $\alpha_i \in A_{n+1}$ and $\beta_i \in S_{n+1} \setminus A_{n+1}$. Now consider the following strategy :

Each P_i chooses α_i. This gives a maximum winning probability $\frac{1}{2}$.

Here are the reasons.

- If given $\sigma \in A_{n+1}$, they win and otherwise lose.
  Since $|A_{n+1}| = \frac{1}{2}|S_{n+1}|$, the winning probability is $\frac{1}{2}$.
- $Pr(\text{They win}) \leq Pr(P_1 \text{ correctly guesses}) = \frac{1}{2}$ since two undetermined locations for P_1 are uniformly distributed. This guarantees that $\frac{1}{2}$ is the maximum value.