05 Let $X \in \mathbb{R}^{n\times n}$ be a symmetric matrix with eigenvalues λ_i and orthonormal eigenvectors $\mathbf{u}_i$. The spectral decomposition gives

$$X = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i \mathbf{u}_i^\top.$$

For a function $f : \mathbb{R} \to \mathbb{R}$, define

$$f(X) := \sum_{i=1}^{n} f(\lambda_i) \mathbf{u}_i \mathbf{u}_i^\top.$$

Let X, $Y \in \mathbb{R}^{n\times n}$ be symmetric. Is it always true that $e^{X+Y} = e^X e^Y$? If not, under what conditions does the equality hold?

Solution. X and Y can be simultaneously diagonalized by an orthogonal matrix if and only if they commute. ([1], Theorem 5 in Chapter 4) From $X = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i \mathbf{u}_i^\top$ and $Y = \sum_{i=1}^{n} \mu_i \mathbf{u}_i \mathbf{u}_i^\top$, we have

$$\begin{aligned}
e^X e^Y &= \sum_{i=1}^{n} e^{\lambda_i} \mathbf{u}_i \mathbf{u}_i^\top \sum_{j=1}^{n} e^{\mu_j} \mathbf{u}_j \mathbf{u}_j^\top \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} e^{\lambda_i + \mu_j} \mathbf{u}_i \mathbf{u}_i^\top \mathbf{u}_j \mathbf{u}_j^\top \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} e^{\lambda_i + \mu_j} \mathbf{u}_i \delta_{ij} \mathbf{u}_j^\top \\
&= \sum_{i=1}^{n} e^{\lambda_i + \mu_i} \mathbf{u}_i \mathbf{u}_i^\top \\
&= e^{X+Y}.
\end{aligned}$$

If X and Y do not commute, the equality does not necessarily hold. Here is a counterexample: Consider

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

and

$$Y = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that

$$XY = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \neq YX = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix};$$

X and Y do not commute.

Then

$$e^X = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^2+1 & e^2-1 \\ e^2-1 & e^2+1 \end{bmatrix}$$

and

$$e^Y = \begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix},$$

so

$$e^X e^Y = \frac{1}{2} \begin{bmatrix} e\left(e^2+1\right) & e^2\left(e^2-1\right) \\ e\left(e^2-1\right) & e^2\left(e^2+1\right) \end{bmatrix}.$$

On the other hand,

$$X + Y = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} -\sqrt{5+\sqrt{5}} & \sqrt{5-\sqrt{5}} \\ \sqrt{5-\sqrt{5}} & \sqrt{5+\sqrt{5}} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 5-\sqrt{5} & 0 \\ 0 & 5+\sqrt{5} \end{bmatrix} \frac{1}{\sqrt{10}} \begin{bmatrix} -\sqrt{5+\sqrt{5}} & \sqrt{5-\sqrt{5}} \\ \sqrt{5-\sqrt{5}} & \sqrt{5+\sqrt{5}} \end{bmatrix},$$

so

$$e^{X+Y} = \frac{1}{10} \begin{bmatrix} -\sqrt{5+\sqrt{5}} & \sqrt{5-\sqrt{5}} \\ \sqrt{5-\sqrt{5}} & \sqrt{5+\sqrt{5}} \end{bmatrix} \begin{bmatrix} e^{\frac{5-\sqrt{5}}{2}} & 0 \\ 0 & e^{\frac{5+\sqrt{5}}{2}} \end{bmatrix} \begin{bmatrix} -\sqrt{5+\sqrt{5}} & \sqrt{5-\sqrt{5}} \\ \sqrt{5-\sqrt{5}} & \sqrt{5+\sqrt{5}} \end{bmatrix}.$$

Obviously $e^{X+Y} \neq e^X e^Y$. ■

References

[1] Richard Bellman. *Introduction to matrix analysis.* SIAM, 1997.