POW 2025-04

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We first claim that condition (a) implies that either $P = 0$ or $d = \deg(P)$. If $P = 0$ then there is nothing to show, so we may assume that $P \neq 0$. Let $n = \deg(P)$ so that $n \geq 0$. Then we can decompose P into a sum $P = P_0 + P_1 + \cdots + P_n$ where each P_i is a polynomial whose terms are exactly of degree i. Notice that for each $i = 0, 1, \ldots, n$, it holds that $P_i(t\boldsymbol{x}) = t^i P_i(\boldsymbol{x})$, and thus

$$t^d P(\boldsymbol{x}) = P(t\boldsymbol{x}) = P_0(t\boldsymbol{x}) + \cdots + P_n(t\boldsymbol{x}) = P_0(\boldsymbol{x}) + tP_1(\boldsymbol{x}) + \cdots + t^n P_n(\boldsymbol{x}).$$

Considering this equation as an identity between polynomials of t, with the fact that $P_n \neq 0$ as $n = \deg(P)$, we conclude that $d = n$.

Now let us consider condition (b). Write P as a sum of monomials $P(\boldsymbol{x}) = \sum_{\boldsymbol{k}} c_{\boldsymbol{k}} x_0^{k_0} x_1^{k_1} \ldots x_5^{k_5}$. From having $P(\boldsymbol{x}) = 0$ if $x_0 = x_1$, it holds that

$$\begin{aligned} P(x_0, x_1, \ldots, x_5) &= P(x_0, x_1, \ldots, x_5) - P(x_1, x_1, \ldots, x_5) \\ &= \sum_{\boldsymbol{k}} \left(x_0^{k_0} - x_1^{k_0}\right) c_{\boldsymbol{k}} x_1^{k_1} \ldots x_5^{k_5} \\ &= \sum_{k_0 \geq 1} \left(x_0^{k_0} - x_1^{k_0}\right) c_{\boldsymbol{k}} x_1^{k_1} \ldots x_5^{k_5} \\ &= (x_0 - x_1) \sum_{k_0 \geq 1} \left(x_0^{k_0 - 1} + x_0^{k_0 - 2} x_1 + \cdots + x_1^{k_0 - 1}\right) c_{\boldsymbol{k}} x_1^{k_1} \ldots x_5^{k_5}. \end{aligned}$$

In particular, this shows that there exists a polynomial $Q_1(\boldsymbol{x})$ such that $P(\boldsymbol{x}) = (x_0 - x_1)Q_1(\boldsymbol{x})$. Here, for any $(i, j) \neq (0, 1)$ with $i < j$, because having $x_i = x_j$ implies $P(\boldsymbol{x}) = 0$ while $x_0 - x_1 \neq 0$, we must have $Q_1(\boldsymbol{x}) = 0$. Thus, we can apply the same procedure on Q_1 to conclude that there exists a polynomial Q_2 such that $Q_1(\boldsymbol{x}) = (x_0 - x_2)Q_2(\boldsymbol{x})$, and hence $P(\boldsymbol{x}) = (x_0 - x_1)(x_0 - x_2)Q_2(\boldsymbol{x})$. Repeating this for all $i < j$, it follows that there exists a polynomial Q such that

$$P(\boldsymbol{x}) = Q(\boldsymbol{x}) \prod_{i<j} (x_i - x_j).$$

As $\prod_{i<j}(x_i - x_j)$ is of degree 15, we either have $P = 0$, which is when $Q = 0$, or $\deg(P) \geq 15$. But we also already know that if $P \neq 0$ then $\deg(P) = d \leq 15$ from condition (a). This leads us to the conclusion that if it were not the case where $P = 0$, we have $\deg(P) = d = 15$, which moreover implies that Q is in fact a constant.

Therefore, a polynomial P satisfies the given conditions if and only if $P(\boldsymbol{x}) = C \prod_{i<j}(x_i - x_j)$ for some constant $C \in \mathbb{R}$.