POW 2025-03: Distinct sums under shifts

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Theorem) (Erdős - Szekeres theorem) Let $r, s \geq 1$ be any integer, and $\{a_i\}$ be sequence of real numbers of length $(r-1)(s-1)+1$. Then $\{a_i\}$ contains nondecreasing subsequence of length r or nonincreasing subsequence of length s.

Proof) Let x_i be the length of the longest nondecreasing subsequence ending with a_i, and y_i be the length of the longest nonincreasing subsequence ending with a_i.

Then every pair (x_i, y_i) are distinct: Let $i < j$. If $a_i \leq a_j$, then $x_i < x_j$ (just add a_j to the end of nondecreasing subsequence of length x_i ends with a_i). Similarly, if $a_i \geq a_j$, then $y_i < y_j$.

Number of possible pair (x_i, y_i) with $1 \leq x_i \leq r-1, 1 \leq y_i \leq s-1$ is $(r-1)(s-1)$, so there exists some (x_i, y_i) with $x_i \geq r$ or $y_i \geq s$. This proves the theorem. ■

Note) Assume $1 < i_1 < \cdots < i_k < n$.

If $a_{i_1}, \cdots, a_{i_k}$ is nondecreasing subsequence, then $a_1 + a_{i_1} + (i_1 - 1), \cdots, a_1 + a_{i_k} + (i_k - 1)$ is increasing. (If $x < y$, then $a_1 + a_{i_x} + (i_x - 1) \leq a_1 + a_{i_y} + (i_x - 1) < a_1 + a_{i_y} + (i_y - 1)$)

Also, if $a_{i_1}, \cdots, a_{i_k}$ is nonincreasing subsequence, then $a_{i_1} + a_n + (n - i_1), \cdots, a_{i_k} + a_n + (n - i_k)$ is decreasing. (If $x < y$, then $a_{i_x} + a_n + (n - i_x) \geq a_{i_y} + a_n + (n - i_x) > a_{i_y} + a_n + (n - i_y)$)

Let $b = \left\lceil \frac{1}{2} n^{1/3} \right\rceil$.

Claim 1) Sequence of length $m+1$ has monotone subsequence of length b.

Proof) By theorem, sequence of length $(b-1)^2 + 1$ has monotone subsequence of length b. Now it is sufficient to prove $m + 1 \geq (b-1)^2 + 1$.

As $\lceil x \rceil < x + 1$, $(b-1)^2 + 1 < \left(\frac{1}{2} n^{1/3} \right)^2 + 1 = \frac{1}{4} n^{2/3} + 1$ holds.

As $(b-1)^2 + 1$ is integer, $(b-1)^2 + 1 \leq \left\lfloor \frac{1}{4} n^{2/3} + 1 \right\rfloor = m + 1$. ■

Claim 2) Statement holds if $1 + \left(\frac{1}{4}n^{2/3} + 1\right)(2n^{1/3} + 1) \le n$.

Proof) Note that $m + 1 \le \frac{1}{4}n^{2/3} + 1$ and $4b - 3 < 4 \cdot \left(\frac{1}{2}n^{1/3} + 1\right) - 3 = 2n^{1/3} + 1$, so

$2 + (m+1)(4b-3) < 2 + \left(\frac{1}{4}n^{2/3} + 1\right)(2n^{1/3} + 1) \le n + 1$. As $2 + (m+1)(4b-3) < n+1$ and $2 + (m+1)(4b-3), n+1$ are integers, $2 + (m+1)(4b-3) \le n$ holds.

Consider $a_1, \cdots, a_{2+(m+1)(4b-3)}$, and split it into subsequences of length $1, m+1, \cdots, m+1, 1$, i.e. $a_1/a_2, \cdots, a_{m+2}/a_3, \cdots, a_{m+3}/\cdots/a_{2+(m+1)(4b-4)}, \cdots, a_{2+(m+1)(4b-3)-1}/a_{2+(m+1)(4b-3)}$.

Then for each subsequence of length $m+1$, there exists monotone subsequence of length b, by **Claim 1**. Let $\{b_{x,j}\}$ be jth element in xth subsequence of length $m+1$.

Considering $\{b_{x,j}\}$ with x odd, there are total $2b-1$ monotone subsequences of length b. By pigeonhole principle, there exists at least b nondecreasing subsequences or at least b nonincreasing subsequences.

Case 1) Assume there are b nondecreasing subsequences. Let $\{a_{i_k}\}$ be union of all such subsequences. Then $a_1 + a_{i_1} + (i_1 - 1), \cdots, a_1 + a_{i_k} + (i_k - 1)$ is increasing:

By **Note**, it is true within each subsequences.

So it is enough to prove between distinct subsequences.

Assume $x < y$ with x, y odd. Note that $b_{x,j_1} = a_{1+(m+1)(x-1)+j_1}$ and $b_{y,j_2} = a_{1+(m+1)(y-1)+j_2}$.

Then $a_{1+(m+1)(x-1)+j_1} - a_{1+(m+1)(y-1)+j_2} \le m$ and

$(1 + (m+1)(x-1) + j_2) - (1 + (m+1)(y-1) + j_2)$

$= ((m+1)(y-1) + j_2) - ((m+1)(x-1) + j_1) \ge 2(m+1) + j_2 - j_1 > m+1$,

so $a_1 + a_{1+(m+1)(x-1)+j_1} + (1 + (m+1)(x-1) + j_1 - 1)$

$< a_1 + a_{1+(m+1)(y-1)+j_2} + (1 + (m+1)(y-1) + j_2 - 1)$.

As $\{a_{i_k}\}$ has length $\ge b^2 = \left\lceil \frac{1}{2}n^{1/3} \right\rceil^2 \ge \frac{1}{4}n^{2/3} \ge m$, statement holds.

Case 2) Assume there are b nonincreasing subsequences. Let $\{a_{i_k}\}$ be union of all such subsequences. Then $a_{i_1} + a_{2+(m+1)(4b-3)} + (2 + (m+1)(4b-3) - i_1), \cdots,$ $a_{i_k} + a_{2+(m+1)(4b-3)} + (2 + (m+1)(4b-3) - i_k)$ is decreasing: We can prove similarly with **Case 1**. ■

Claim 3) Statement holds for all $n \geq 1$.

If $m = 1$, then statement is trivial. So assume $m \geq 2$, $\frac{1}{4}n^{2/3} \geq 2$, $n^{1/3} \geq 2\sqrt{2}$.

$1 + \left(\frac{1}{4}n^{2/3} + 1\right)(2n^{1/3} + 1) \leq n$ gives $2n - n^{2/3} - 8n^{1/3} - 8 \geq 0$.

Consider function $f(x) = 2x^3 - x^2 - 8x - 8$. Then $f(2\sqrt{2}) = 16\sqrt{2} - 16 > 0$ and

$f'(x) = 6x^2 - 2x - 8 = 6(x+1)(x - 4/3) > 0$ for $x \geq 2\sqrt{2}$, so $f(x) > 0$ for $x \geq 2\sqrt{2}$.

Put $x = n^{1/3}$, then we get desired result. ■