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Theorem 1. *Let X and Y be closed manifolds, and suppose X is a finite-sheeted cover of Y. If $b_1(Y) > 0$, then $b_1(X) > 0$.*

Proof. Let
$$p\colon X \longrightarrow Y$$
be a covering map with finite degree; in other words, p is a finite-sheeted cover. By definition, $b_1(\,\cdot\,)$ denotes the first Betti number over $\mathbf{Q}$, so
$$b_1(Y) \;=\; \dim_{\mathbf{Q}} H^1(Y;\mathbf{Q}).$$
Equivalently, we can interpret $b_1(Y)$ as the rank of the abelianization of $\pi_1(Y)$:
$$b_1(Y) \;=\; \dim_{\mathbf{Q}}\Big(\pi_1(Y)_{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbf{Q}\Big),$$
where
$$\pi_1(Y)_{\mathrm{ab}} \;=\; \frac{\pi_1(Y)}{[\pi_1(Y), \pi_1(Y)]}.$$

Since $b_1(Y) > 0$, there exists a nontrivial homomorphism
$$\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}.$$
Indeed, having $b_1(Y) > 0$ means $\pi_1(Y)$ admits a surjection onto an infinite cyclic group $\mathbb{Z}$, signifying a free $\mathbb{Z}$-summand in its abelianization.

On the other hand, the finite-sheeted covering $p\colon X \to Y$ implies that
$$\pi_1(X) \;\subset\; \pi_1(Y)$$
is a finite-index subgroup. Suppose, for the sake of contradiction, that this entire subgroup $\pi_1(X)$ were contained in the kernel of φ, i.e.,
$$\pi_1(X) \;\subseteq\; \ker(\varphi).$$
Since φ is surjective onto $\mathbb{Z}$ (an infinite group), its kernel $\ker(\varphi)$ must have infinite index in $\pi_1(Y)$. But $\pi_1(X)$ is a *finite*-index subgroup of $\pi_1(Y)$. A finite-index subgroup cannot be contained entirely in an infinite-index subgroup. This is the key contradiction.

Hence $\pi_1(X)$ must *not* lie in $\ker(\varphi)$; therefore,
$$\varphi|_{\pi_1(X)} :\; \pi_1(X) \;\longrightarrow\; \mathbb{Z}$$
is a nontrivial homomorphism. Nontriviality means $\pi_1(X)$ also surjects (perhaps onto a subgroup of $\mathbb{Z}$, but still infinite), which yields a free $\mathbb{Z}$-summand in its abelianization. Consequently,
$$b_1(X) \;=\; \dim_{\mathbf{Q}}\Big(\pi_1(X)_{\mathrm{ab}} \otimes \mathbf{Q}\Big) \;>\; 0.$$
Thus, X inherits a nonzero first Betti number from Y. This proves the statement. ■