# POW 2025-01 INTEGER SOLUTIONS OF RECIPROCALS

JIHYEOK JEONG

**Problem.** *Find all positive integers* $a$, $n$ *such that*

$$\frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n}$$

*is an integer.*

***Solution.*** There exists no such positive integers $a$, $n$. Suppose there exists $a, n$ such that

$$\frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n} = k$$

for some integer $k$. For each $j = 0, 1, \cdots, n$ define $N_j = a(a+1)\cdots(a+n)/(a+j)$. By multiplying $a(a+1)\cdots(a+n)$ on both sides we get

$$N_0 + N_1 + \cdots + N_n = ka(a+1)\cdots(a+n). \tag{1}$$

Since any $n+1$ consecutive integers form a complete system of residues modulo $n+1$, there is one and only one integer $i$, $0 \le i \le n$ such that $a + i \equiv 0 \pmod{n+1}$. Hence for each $j$, $N_j \equiv 0 \pmod{n+1}$ if $j \ne i$ and $N_j \not\equiv 0 \pmod{n+1}$ if $j = i$. Taking modulo $n+1$ on both sides of (1) gives us a contradiction. (Note that $ka(a+1)\cdots(a+n) \equiv 0 \pmod{n+1}$.)