POW 2025-01 Integer sum of reciprocals

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Claim. There are no positive integers a, n such that

$$\sum_{j=a}^{a+n} \frac{1}{j} \in \mathbb{Z}$$

Proof. Suppose that $a, n \in \mathbb{Z}_{>0}$ satisfy

$$m := \sum_{j=a}^{a+n} \frac{1}{j} \in \mathbb{Z}$$

Let k be the largest integer such that $2^k \leq a + n$.

If $2^k < a$, then $n = (a+n) - a < 2^{k+1} - 2^k = 2^k < a$, so

$$0 < m = \sum_{j=a}^{a+n} \frac{1}{j} < \sum_{j=a}^{a+n} \frac{1}{a} = \frac{n+1}{a} \leq 1$$

Contradiction.

If $2^k \geq a$, then let $S = \{a, a+1, \ldots, a+n\} \setminus \{2^k\}$ and $l = \mathrm{lcm}(S)$. Note that $2^k \nmid l$ since there is no integer multiple of 2^k in S. Then

$$m = \sum_{j=a}^{a+n} \frac{1}{j} = \sum_{j \in S} \frac{1}{j} + \frac{1}{2^k}$$

$$\frac{l}{2^k} = lm - \sum_{j \in S} \frac{l}{j} \in \mathbb{Z}$$

Contradiction. □