POW 2024-21: The Realizability of Fundamental Group Homomorphisms

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Theorem 1) By regarding loops as singular 1-cycles, we obtain a homomorphism $h : \pi_1(X, x_0) \to H_1(X)$. If X is path-connected, then h is surjective and has kernel the commutator subgroup of $\pi_1(X)$, so h induces an isomorphism from the abelianization of $\pi_1(X)$ onto $H_1(X)$.

Proof) Allen Hatcher, Algebraic Topology, Theorem 2A.1.

Theorem 2) (Universal coefficient theorem for cohomology)

If a chain complex C of free abelian groups has homology groups $H_n(C)$, then

$H^n(C; G) \cong \text{Ext}(H_{n-1}(C), G) \oplus \text{Hom}(H_n(C), G)$.

Proof) Allen Hatcher, Algebraic Topology, Theorem 3.2..

Theorem 3) $H^*(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$, where α is generator of $H^1(\mathbb{R}P^n; \mathbb{Z}_2)$.

Proof) Allen Hatcher, Algebraic Topology, Theorem 3.12.

Note that for $n \geq 2$, $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$ and $H_0(\mathbb{R}P^n) \cong \mathbb{Z}$.

Claim) For any continuous map $f : \mathbb{R}P^3 \to \mathbb{R}P^2$, induced homomorphism

$f^* : \pi_1(\mathbb{R}P^3) \to \pi_1(\mathbb{R}P^2)$ is trivial.

Proof) Assume not. Then f^* should be isomorphism, since $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$.

By theorem 1, f^* induces isomorphism $f_1 : H_1(\mathbb{R}P^3) \to H_1(\mathbb{R}P^2)$.

By theorem 2, f_1 induces isomorphism $f_2 : H^1(\mathbb{R}P^2; \mathbb{Z}_2) \to H^1(\mathbb{R}P^3; \mathbb{Z}_2)$.

($H_0(\mathbb{R}P^n) \cong \mathbb{Z}$, which is free, so $\text{Ext}(H_0(\mathbb{R}P^n), \mathbb{Z}_2) = 0$)

Let α and β be generator of $H^1(\mathbb{R}P^2; \mathbb{Z}_2)$ and $H^1(\mathbb{R}P^3; \mathbb{Z}_2)$, respectively.

As $H^1(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ for $n \geq 2$, $f_2(\alpha) = \beta$ is only possible case.

But by theorem 3, $\alpha^3 = 0$ and $\beta^3 \neq 0$. Then $0 = f_2(\alpha^3) = \beta^3 \neq 0$, so it is contradiction. ■

Counterexample) Let $X = \mathbb{R}P^2 \times \mathbb{R}P^3$. Then $\pi_1(X) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle$.

Let $\phi : \pi_1(X) \to \pi_1(X)$ by $a \mapsto b$ and $b \mapsto a$, which is automorphism.

But there is no continuous map $X \to X$ that induces $b \mapsto a$, by claim.