**20** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that the sequence $f(x)$, $f(2x)$, $f(3x)$, $\cdots$ converges to 0 for any $x > 0$. Prove or disprove that

$$\lim_{x\to\infty} f(x) = 0.$$

*Solution.* The proposition is a famous application of the **Baire category theorem**, which we will take for granted.

**Definition.** A space $X$ is said to be a ***Baire space*** if the following condition holds: Given any countable collection $\{A_n\}$ of closed sets of $X$ each of which has empty interior in $X$, their union $\bigcup A_n$ also has empty interior in $X$.

**Theorem** (Baire category theorem)**.** *Every locally compact Hausdorff space is a Baire space.*

Since $f(x)$ is continuous, so is $f_n(x) \coloneqq f(nx)$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$ be given and define

$$E_N = \{x \mid n \geq N \implies |f_n(x)| \leq \epsilon\} = \bigcap_{n\geq N} f_n^{-1}([-\epsilon, \epsilon])$$

for each $N \in \mathbb{N}$. Note that each $E_N$ is closed by continuity of $f_n$. Since $\lim_{n\to\infty} f_n(x) = 0$ for all $x > 0$, we can write $\mathbb{R}^+ = \bigcup_{N=1}^{\infty} E_N$.

Because $\mathbb{R}^+$ is a Baire space by the theorem, should each $E_N$ have empty interior, so does $\mathbb{R}^+$, which is not the case. Hence, some $E_{N_0}$ must contain an open interval, say $(a, b)$. Thus,

$$n \geq N_0 \implies f_n((a, b)) = f((na, nb)) \subset [-\epsilon, \epsilon].$$

Take a sufficiently large integer $M > \max\{N_0, a/(b-a)\}$ such that $n \geq M \implies n(b-a) > na \implies (n+1)a < nb$. Finally, we have

$$x \in \bigcup_{n\geq M} (na, nb) = (Ma, \infty) \implies |f(x)| \leq \epsilon.$$

Therefore, we conclude that $\lim_{x\to\infty} f(x) = 0$. □



Typeset by LaTeX