POW 2024-19: Stationary function

수리과학과 석박통합과정 김준홍

Let $h(x) = 2024g(x)$. Then $h(t) : [0, +\infty) \to [0, +\infty)$ is a decreasing continuous function with $h(0) = 2024$. Also, for every $s, t \geq 0$,

$t^{11}g(s+t) \leq 2024[g(s)]^2 \Leftrightarrow t^{11}2024g(s+t) \leq 2024^2[g(s)]^2 \Leftrightarrow t^{11}h(s+t) \leq [h(s)]^2$.

Put $s = 0$, then $t^{11}h(t) \leq 2024^2$, so $h(t) \leq \dfrac{2024^2}{t^{11}}$.

Let $c > 1$ be arbitrary constant, and $A = \sqrt[11]{c}$. Let $S_0 = \sqrt[11]{2024^2}A^2$.

Then $g(S_0) \leq \dfrac{2024^2}{(\sqrt[11]{2024^2}A^2)^{11}} = \dfrac{2024^2}{2024^2c^2} = c^{-2}$. Let $S_n = S_0 + \displaystyle\sum_{k=1}^{n} \frac{1}{A^n}$.

Claim) $g(S_n) \leq c^{-(n+2)}$ for all $n \geq 0$.

Proof) Use induction on n. For $n = 0$, it clearly holds. Assume it holds for n.

Put $t = \dfrac{1}{A^{n+1}}$ and $s = S_n$, then we get $\left(\dfrac{1}{A^{n+1}}\right)^{11} h(S_{n+1}) \leq [h(S_n)]^2$.

So $h(S_{n+1}) \leq (A^{n+1})^{11}[h(S_n)]^2 = c^{n+1}[h(S_n)]^2 \leq c^{n+1}(c^{-(n+2)})^2 = c^{-(n+3)}$.

So it also holds for $n + 1$. ■

Note that $S := \displaystyle\lim_{n\to\infty} S_n = S_0 + \frac{1/A}{1 - 1/A} = S_0 + \frac{1}{A-1} = \sqrt[11]{2024^2}A^2 + \frac{1}{A-1}$.

As $g(S) \leq g(S_n) \leq c^{-(n+2)}$ for any n, $g(S) = 0$.

Choose $A = 1.4$, then $S = \sqrt[11]{2024^2}1.4^2 + \dfrac{1}{1.4 - 1} < 2\sqrt[11]{2048^2} + 2.5 = 2 \times 4 + 2.5 = 10.5 < 11$.

Therefore, $0 \leq g(12) \leq g(11) \leq g(S) = 0$, so $g(11) = g(12) = 0$.