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THE NONNEGATIVE TRIPLE SEQUENCE CHALLENGE

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Problem. Let $f(n)$ denote the number of possible sequences of length n, where each term is either $0, 1$, or -1, such that the product of every three consecutive numbers is nonnegative. Compute $f(33)$.

Solution. For $n \geq 3$, say a sequence in $\{0, 1, -1\}$ is an *n-triple* if it has length n, and the product of every three consecutive numbers is nonnegative. Then $f(n)$ becomes the number of distinct n-triples. Now, we make a few observations.

(i) The only possibilities that three consecutive terms multiply to give a negative number are

$$(-1, -1, -1), (1, 1, -1), (1, -1, 1), (-1, 1, 1).$$

Thus, a sequence of length n is an n-triple if and only if it contains none of the four possibilities above.

(ii) The first n terms of any $(n+1)$-triple forms an n-triple. In other words, every $(n+1)$-triple is obtained by taking an n-triple, then adding an $(n+1)$th term.

(iii) Conversely, taking an n-triple and adding an $(n+1)$th term does not always give an $(n+1)$-triple. For an n-triple $(a_k)_{k=1}^{n}$, the following are exactly the ways to add the $(n+1)$th term to obtain an $(n+1)$-triple.

$$\begin{cases} \text{If } (a_k) \text{ ends with } 1, 1 \text{ or } -1, -1\text{: We can let } a_{n+1} = 0 \text{ or } 1. \\ \text{If } (a_k) \text{ ends with } 1, -1 \text{ or } -1, 1\text{: We can let } a_{n+1} = 0 \text{ or } -1. \\ \text{Otherwise: We can let } a_{n+1} = 0, 1 \text{ or } -1. \end{cases}$$

For the first two cases, one n-triple generates two $(n+1)$-triples, and for the third case, one n-triple generates three $(n+1)$-triples. Thus, if we let

$$k(n) := \#\ n\text{-triples with } a_{n-1}a_n = 0,$$

then we obtain the recursive relation

$$f(n+1) = 3k(n) + 2(f(n) - k(n)).$$

(iv) Let

$$g(n) := \#\ n\text{-triples with } a_n = 0,$$

$$h(n) := \#\ n\text{-triples with } a_{n-1} = 0 \text{ and } a_n \neq 0.$$

Then $g(n) + h(n) = k(n)$.

(v) For any n-triple, we can let $a_{n+1} = 0$ and generate an $(n+1)$-triple. Therefore, the $(n+1)$-triples with $a_{n+1} = 0$ are precisely all the n-triples extended via adding $a_{n+1} = 0$. $\therefore g(n+1) = f(n)$.

(vi) The $(n+1)$-triples with $a_n = 0$ and $a_{n+1} \neq 0$ are precisely the n-triples with $a_n = 0$ extended via adding $a_{n+1} = 1$ or -1. $\therefore h(n+1) = 2g(n)$.

To sum up, the problem reduces to finding $f(33)$ from the following system of recursive relations:

$$\begin{cases} f(n+1) = 2f(n) + g(n) + h(n) \\ g(n+1) = f(n) \\ h(n+1) = 2g(n), \end{cases}$$

where the initial values $f(3) = 23, g(3) = 9, h(3) = 6$ can be easily found by examining all possible 3-triples. Equivalently, one can write

$$f(n+3) = 2f(n+2) + f(n+1) + 2f(n)$$

with $f(3) = 23, f(4) = 61, f(5) = 163$. Computing this (I used Python to define a recursive function) gives $f(33) = 126963119145083$.

Alternatively, we can derive a generating function of the sequence $f(n)$. First, notice that we can extend f to values $n = 0, 1, 2$ by defining $f(0) = 1, f(1) = 3, f(2) = 9$; from there the relation $f(n+3) = 2f(n+2) + f(n+1) + 2f(n)$ yields $f(3) = 23, f(4) = 61, f(5) = 163$, and so on.

Now, let $F(x) := \sum_{n=0}^{\infty} f(n)x^n$ be the generating function of $f(n)$. Then,

$$\sum_{n=0}^{\infty} f(n+3)x^n = 2\sum_{n=0}^{\infty} f(n+2)x^n + \sum_{n=0}^{\infty} f(n+1)x^n + 2\sum_{n=0}^{\infty} f(n)x^n$$

$$\implies \sum_{n=3}^{\infty} f(n)x^{n-3} = 2\sum_{n=2}^{\infty} f(n)x^{n-2} + \sum_{n=1}^{\infty} f(n)x^{n-1} + 2\sum_{n=0}^{\infty} f(n)x^n$$

$$\implies \frac{1}{x^3}(F(x) - f(0) - f(1)x - f(2)x^2) = \frac{2}{x^2}(F(x) - f(0) - f(1)x) + \frac{1}{x}(F(x) - f(0)) + 2F(x)$$

$$\implies F(x) = \frac{1 + x + 2x^2}{1 - 2x - x^2 - 2x^3}.$$

Therefore, we can alternatively express $f(33)$ as the expansion of $F(x)$, by

$$f(33) = \frac{F^{(33)}(0)}{33!}.$$

Computing this (I used WolframAlpha) gives 126963119145083, the same result as above.

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