17 Suppose that $p(x)$ is a degree n polynomial with complex coefficients such that $p(x) \geq 0$ for any real number x. Prove that

$$p(x) + p'(x) + \cdots + p^{(n)}(x) \geq 0$$

for any real number x.

Solution. If $n = 0$, there is nothing to prove, so we assume $n \geq 1$. Note that n must be even and $\lim_{|x|\to\infty} p(x) = \infty$. Also, we can write $p(x) = p_1(x) + ip_2(x)$, where $p_1, p_2 \in \mathbb{R}[x]$. Since $p(x) \geq 0$ for all $x \in \mathbb{R}$, $p_2(x) = 0$. It suffices to consider $p \in \mathbb{R}[x]$.

Let $f(x) = \sum_{k=0}^{n} p^{(k)}(x)$. Then $f'(x) = \sum_{k=1}^{n} p^{(k)}(x)$, so $f(x) = p(x) + f'(x)$. Observe that $\lim_{|x|\to\infty} f(x) = \infty$ because $\deg(f) = \deg(p) > \deg(f')$; p determines the limiting behavior of f.

Hence, there are real numbers a, b, $M > 0$ with such that $a < b$ and $x \in (-\infty, a) \cup (b, \infty) \implies f(x) > M$. By the extreme value theorem, f attains a minimum $f(c)$ for some $c \in [a, b]$, which is indeed global. Therefore, $f(x) \geq f(c) = p(c) + f'(c) = p(c) \geq 0$, as desired. □