KAIST POW 2024-16
Stay positive!

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Problem. Let $A = [a_{ij}]_{1 \leq i,j \leq 5}$ be a 5×5 positive definite (real) matrix. Show that the matrix $[a_{ij} / (i+j)]$ is also positive definite.

Solution. Let x_1, x_2, x_3, x_4, x_5 be five arbitrary real numbers. It is given that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j > 0 \tag{1}$$

and our claim is that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_{ij}}{i+j} x_i x_j > 0 \tag{2}$$

Observe that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_{ij}}{i+j} x_i x_j = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \int_0^\infty e^{-(i+j)t} dt \tag{3}$$

$$= \int_0^\infty \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i e^{-it} x_j e^{-jt} dt \tag{4}$$

Here (4) comes from the Fubini's theorem. For any fixed $t \in \mathbb{R}$, subtituting $x_i e^{-it}$ in place of x_i in (1), we have

$$f(t) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \left(x_i e^{-it} \right) \left(x_j e^{-jt} \right) > 0 \tag{5}$$

Hence (4) is a integral of positive function $f(t)$, and thus positive. Its convergence is obvious; it was just a finite sum in the beginning. It follows that (2) holds for arbitrary $x_1, \ldots, x_5$, and therefore $[a_{ij} / (i+j)]$ is positive definite. □