POW 2024-15: The Narrow Gap Sequence Conundrum

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Actually, $[2024 \quad 2022 \quad \cdots \quad 8 \quad 6 \quad 3 \quad 1 \quad 4 \quad 2 \quad 5 \quad 7 \quad 9 \quad \cdots \quad 2021 \quad 2023]$ satisfies the condition. But this is too simple, so let me state the proof of generalization about the problem. Let arrangement of $1, 2, \cdots, n$ in a sequence as $n-$seq, by convenience.

Condition A) Difference between any two adjacent numbers is greater than 1 but less than 4.

Condition B) Left end and right end of a $n-$seq is n and $n-1$(or $n-1$ and n).

Claim 1) For $n \geq 4$, there exists $n-$seq that satisfies condition A.

Moreover, if $n \geq 6$, there exists $n-$seq that satisfies both condition A and B.

Proof) $n = 4$: $[3 \quad 1 \quad 4 \quad 2]$, $n = 5 : [3 \quad 1 \quad 4 \quad 2 \quad 5]$

$n \geq 6$, n is odd: $[n-1 \quad n-3 \quad \cdots \quad 6 \quad 3 \quad 1 \quad 4 \quad 2 \quad 5 \quad \cdots \quad n-2 \quad n]$

$n \geq 6$, n is even: $[n \quad n-2 \quad \cdots \quad 6 \quad 3 \quad 1 \quad 4 \quad 2 \quad 5 \quad \cdots \quad n-3 \quad n-1]$

Then these sequences satisfies the claim. ■

Using above claim, we can construct $n-$seq with more condition.

Condition C) Difference between left end and right end of a $n-$seq is greater than 1 but less than 4.

Claim 2) For $n \geq 10$, there exists $n-$seq that satisfies both condition A and C, so that sequence forms a cycle.

Proof) For $n \geq 10$, there is $(n-4)$-seq and 6-seq ($[6 \quad 3 \quad 1 \quad 4 \quad 2 \quad 5]$) that satisfies both condition A and B. Then $[n-5 \quad n-2 \quad n \quad n-3 \quad n-1 \quad n-4]$, which is kind of reflection of $6-$seq, also satisfies both condition A and B. As $(n-4)$-seq has $n-4$ and $n-5$ in both end, we can connect those two sequences. Then that connected sequence satisfies both condition A and C. ■

Remark) Claim 1 and 2 are actually if and only if condition, i.e. only given numbers of n in claim satisfies the condition.