POW 2024-14

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Denote the given sum by

$$S := \sum_{k=0}^{\infty} \frac{1}{(6k+1)(6k+2)(6k+3)(6k+4)(6k+5)(6k+6)}.$$

For positive integers a and b, recall that $B(a,b) = \frac{(a-1)!(b-1)!}{(a+b-1)!}$ where B denotes the beta function. Thus it holds that

$$\begin{aligned} S &= \sum_{k=0}^{\infty} \frac{(6k)!}{(6k+6)!} \\ &= \sum_{k=0}^{\infty} \frac{B(6k+1,6)}{5!} \\ &= \frac{1}{5!} \sum_{k=0}^{\infty} \int_0^1 t^{6k}(1-t)^5 \, \mathrm{d}t. \end{aligned}$$

As $t^{6k}(1-t)^5 \geq 0$ whenever $0 \leq t \leq 1$, by the monotone convergence theorem the order of summation and integration can be exchanged. This leads to

$$\begin{aligned} S &= \frac{1}{5!} \int_0^1 \sum_{k=0}^{\infty} t^{6k}(1-t)^5 \, \mathrm{d}t \\ &= \frac{1}{5!} \int_0^1 \frac{(1-t)^5}{1-t^6} \, \mathrm{d}t \\ &= \frac{1}{5!} \int_0^1 \left(\frac{16}{3} \cdot \frac{1}{1+t} + \frac{1}{6} \cdot \frac{1+t}{1-t+t^2} - \frac{9}{2} \cdot \frac{1+t}{1+t+t^2} \right) \mathrm{d}t \\ &= \frac{1}{5!} \left(\frac{16}{3} \log(1+t) + \frac{1}{12} \log(1-t+t^2) + \frac{\sqrt{3}}{6} \tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) - \frac{9}{4} \log(1+t+t^2) - \frac{3\sqrt{3}}{2} \tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) \right) \Bigg|_0^1 \\ &= \frac{192 \log 2 - 81 \log 3 - 7\pi\sqrt{3}}{4320} \end{aligned}$$

and we are done.