KAIST POW 2024-13
Concave functions (revisited)

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Problem. Let $u_n(t)$, $n = 1, 2, \ldots$ be a sequence of concave functions on $\mathbb{R}$. Let $g(t)$ be a differentiable function on $\mathbb{R}$. Assume $\liminf_{n\to\infty} u_n(t) \geq g(t)$ for every t and $\lim_{n\to\infty} u_n(0) = g(0)$. Suppose $u_n'(0)$ exist for $n = 1, 2, \ldots$. Compare $\lim_{n\to\infty} u_n'(0)$ and $g'(0)$.

Solution. Since u_n are concave on $\mathbb{R}$ and $u_n'(0)$ exists, we have

$$u_n(t) \leq u_n(0) + u_n'(0)t$$

for all $n = 1, 2, \ldots$ and $t \in \mathbb{R}$, whence

$$g(t) \leq \liminf_{n\to\infty} u_n(t) \leq \liminf_{n\to\infty} \big(u_n(0) + u_n'(0)t\big) = g(0) + \liminf_{n\to\infty} \big[u_n'(0)t\big]$$

It follows that

$$g(t) - g(0) - tg'(0) \leq \liminf_{n\to\infty} \big[\big(u_n'(0) - g'(0)\big)\, t\big] \tag{1}$$

Divide both sides of the (1) by t and take $t \searrow 0$ and $t \nearrow 0$ give the following two inequalities.

$$\lim_{t\searrow 0} \frac{g(t) - g(0) - tg'(0)}{t} = 0 \leq \liminf_{n\to\infty} \big[u_n'(0) - g'(0)\big]$$
$$\lim_{t\nearrow 0} \frac{g(t) - g(0) - tg'(0)}{t} = 0 \geq \limsup_{n\to\infty} \big[u_n'(0) - g'(0)\big]$$

Therefore $\lim_{n\to\infty} u_n'(0) = g'(0)$. □