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(**Problem**) Count the number of distinct matrices A, where two matrices are considered identical if one can be obtained from the other by rearranging rows and columns, that have the following properties:

(1) A is a 7×7 matrix and every entry of A is 0 or 1.

(2) Each row of A contains exactly 3 non-zero entries.

(3) For any two distinct rows i and j of A, there exists exactly one column k such that $A_{ik}\neq 0$ and $A_{jk}\neq 0$.

(**Solution**) Notice that the following matrix M satisfies condition (1) ~ (3).

$$M=\begin{bmatrix}1&1&1&0&0&0&0\\1&0&0&1&1&0&0\\1&0&0&0&0&1&1\\0&1&0&1&0&1&0\\0&1&0&0&1&0&1\\0&0&1&1&0&0&1\\0&0&1&0&1&1&0\end{bmatrix}$$

Indeed, M is the only solution. That is, for any matrix satisfying (1) ~ (3), say B, we can rearrange rows and columns of B to obtain M.

Let B be a matrix satisfying (1) ~ (3).

(Notation) $[B]_i$: i-th row of B.

$[B]^j$: j-th column of B.

$B=(b_{i,j})\qquad(1\leq i,k\leq 7)$

$S(N):=$ sum of all entries of matrix N.

(observation 1) No column of B has more than 3 non-zero entries.

(pf) Assume that $[B]^1$ has 4 or more non-zero entries. WLOG, assume $b_{1,1}=b_{2,1}=b_{3,1}=b_{4,1}=1$.
Let C be the 4×6 matrix such that

$$B=\begin{bmatrix}1&&&&&&\\1&&&C&&&\\1&&&&&&\\1&&&&&&\\ *&*&*&*&*&*&*\\ *&*&*&*&*&*&*\\ *&*&*&*&*&*&*\end{bmatrix}.$$

Then, by condition (3), every column of C has at most 1 non-zero entry, thus C has at most 6 non-zero entries. But, by condition (2), C must have 8 non-zero entries. Contradiction!

$\therefore$ $[B]^1$ has at most 3 non-zero entries.

Similarly, other columns of B also have at most 3 non-zero entries. ∎

(observation2) Every column of B has 3 non-zero entries.

(pf) Assume that some column of B does not have 3 non-zero entries.

Then, $21 = \sum_{i=1}^{7} 3 = \sum_{i=1}^{7} S([B]_i) = S(B) = \sum_{j=1}^{7} S([B]^j) < \sum_{j=1}^{7} 3 = 21$. Contradiction. ∎

Since each row and column of B has exactly 3 non-zero entries, we can rearrange B and obtain the following B' where D, E, F, G are 2×2 matrices.

$$B \sim B' = \begin{bmatrix} 1\,1\,1\,0\,0\,0\,0 \\ 1\,0\,0\,1\,1\,0\,0 \\ 1\,0\,0\,0\,0\,1\,1 \\ 0\,1\,0 \quad D \quad E \\ 0\,1\,0 \\ 0\,0\,1 \quad F \quad G \\ 0\,0\,1 \end{bmatrix}$$

(observation 3) $D, E, F, G \in \left\{ \begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix}, \begin{pmatrix} 0\,1 \\ 1\,0 \end{pmatrix} \right\}$

(pf) Notice that $S(D), S(E), S(F), S(G) < 3$ and $S(D) + S(E) + S(F) + S(G) = 8$.

Then $S(D) = S(E) = S(F) = S(G) = 2$.

Since D, E, F, G can not have two non-zero entries in the same row or column, we get the desired result. ∎

If $D = \begin{pmatrix} 0\,1 \\ 1\,0 \end{pmatrix}$, we rearrange B' by exchanging $[B']^4$ and $[B']^5$.

Similarly, if $E = \begin{pmatrix} 0\,1 \\ 1\,0 \end{pmatrix}$, we rearrange B' by exchanging $[B']^6$ and $[B']^7$.

Also, if $F = \begin{pmatrix} 0\,1 \\ 1\,0 \end{pmatrix}$, we rearrange B' by exchanging $[B']_6$ and $[B']_7$.

Now, we have the following form.

$$B \sim B' \sim B'' = \begin{bmatrix} 1\,1\,1\,0\,0\,0\,0 \\ 1\,0\,0\,1\,1\,0\,0 \\ 1\,0\,0\,0\,0\,1\,1 \\ 0\,1\,0\,1\,0\,1\,0 \\ 0\,1\,0\,0\,1\,0\,1 \\ 0\,0\,1\,1\,0 \quad G \\ 0\,0\,1\,0\,1 \end{bmatrix}$$

If $G = \begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix}$, then the 4th and 6th rows does not satisfy condition(3).

$\therefore\ G = \begin{pmatrix} 0\,1 \\ 1\,0 \end{pmatrix}$ and we are done.

$$B \sim B' \sim B'' = \begin{bmatrix} 1\,1\,1\,0\,0\,0\,0 \\ 1\,0\,0\,1\,1\,0\,0 \\ 1\,0\,0\,0\,0\,1\,1 \\ 0\,1\,0\,1\,0\,1\,0 \\ 0\,1\,0\,0\,1\,0\,1 \\ 0\,0\,1\,1\,0 \quad G \\ 0\,0\,1\,0\,1 \end{bmatrix} = \begin{bmatrix} 1\,1\,1\,0\,0\,0\,0 \\ 1\,0\,0\,1\,1\,0\,0 \\ 1\,0\,0\,0\,0\,1\,1 \\ 0\,1\,0\,1\,0\,1\,0 \\ 0\,1\,0\,0\,1\,0\,1 \\ 0\,0\,1\,1\,0\,0\,1 \\ 0\,0\,1\,0\,1\,1\,0 \end{bmatrix} = M.$$