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Assume f(x) be such polynomial of degree n. (i.e. if f(x) is rational, x is also a rational number) Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$.

Claim 1 : Every coefficient a_i of f(x) must be rational.

proof : Since the image of f(x) is infinite, we can choose n+1 distinct rational numbers $r_1, r_2, \cdots, r_{n+1}$, such that $f(x) - r_i = 0$ has at least one solution. Let such solution α_i, i.e. $f(\alpha_i) = r_i$. Note that each α_i must be rational number by our assumption on $f(x)$.

Consider the matrix

$$A = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & ... & \alpha_1^n \\ 1 & \alpha_2 & \alpha_2^2 & ... & \alpha_2^n \\ & & ... & & \\ & & & & \\ 1 & \alpha_{n+1} & \alpha_{n+1}^2 & ... & \alpha_{n+1}^n \end{bmatrix}$$

Let $v = [a_0, a_1, \cdots, a_n]^T$ and $r = [r_1, r_2, \cdots, r_{n+1}]$. Then $Av = r$ holds.

Here, $|det(A)|$=$|\Pi_{i \neq j}(\alpha_i - \alpha_j)|$(Lemma 1). Since $f(\alpha_i) = r_i \neq r_j = f(\alpha_j)$ by our selection of r, $det(A) \neq 0$. Finally, $v = A^{-1}r$. Since every component of A and r is rational, so must be the components of v. ■

Claim 2 : Polynomial that satisfies $'$if f(x) is rational, x is also a rational number$'$ can have degree at most 1.

Proof : Let $|a_i| = p_i/q_i$ for coprime integer p_i and q_i. Also, WLOG $a_n > 0$. f' goes to infinity as x increases. For M_1 large enough, $f(x)$ is strictly increasing on $x > M_1$, so we can define $y = f^{-1}(x) = g(x)$ on $x > f(M_1), y > M_1$.

Assume that f has degree at least 2. Then f' goes to infinity, so g' goes to 0. Therefore, there is some $M_2 > f(M_1)$ such that $g(x+1) - g(x) = g'(c_x) < 1/p_n q_0$ whenever $x > M_2$. ($p_n \neq 0$, and q_0 is defined 1 if $p_0 = 0$)

Let $N > M_2$ be some positive integer. Since $f(g(N)) = N, f(g(N+1)) = N+1$,

$g(N), g(N+1)$ must be rationals. Notice that every coefficient of f is rational by claim 1. Therefore, by the rational root theorem, (after abbreviation)denominator of $g(N+1)$ and $g(N)$ must be divisor of $p_n q_0$. Especially, $g(N+1) - g(N) \geq 1/p_n q_0$. It is contradiction to the definition of M_2. Now, f must be linear. ■

To sum up the claim 1,2, f must be a linear function with rational coefficients. Let $f(x) = ax + b$, when a, b are rational numbers. If $a \neq 0$, $f(x) = q$ implies $x = (q - b)/a$, which is a rational number. When $a = 0$, constant function satisfies the condition only if it is constant to some irrational number. Therefore the answer is "Every degree 1, nonconstant polynomial with rational coefficients or irrational constant function".

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(Lemma 1) $|det(A)|$=$|\Pi_{i \neq j}(\alpha_i - \alpha_j)|$

proof : Use induction on n. When $n = 1$, $det(A) = \alpha_2 - \alpha_1$, so the claim holds.

Now consider

$$A(x) = \begin{bmatrix} 1 & x & x^2 & ... & x^n \\ 1 & \alpha_2 & \alpha_2^2 & ... & \alpha_2^n \\ & & ... & & \\ & & & & \\ 1 & \alpha_{n+1} & \alpha_{n+1}^2 & ... & \alpha_{n+1}^n \end{bmatrix}$$

$det(A(x))$ is a polynomial of degree n, with $det(A(\alpha_i)) = 0$ for $i \in \{2, 3, \cdots, n+1\}$. Therefore, $det(A(x)) = C_1(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_{n+1})$. Note that x^n coefficient of $det(A(x))$ is $(-1)^n det(A_{n-1})$, when

$$A_{n-1} = \begin{bmatrix} 1 & \alpha_2 & \alpha_2^2 & ... & \alpha_2^{n-1} \\ & & ... & & \\ & & & & \\ 1 & \alpha_{n+1} & \alpha_{n+1}^2 & ... & \alpha_{n+1}^{n-1} \end{bmatrix}$$

By the induction hypothesis, $|C_1| = |det(A_{n-1})| = |\Pi_{i \neq j \in \{2,3,\cdots,n+1\}}(\alpha_i - \alpha_j)|$. After putting α_1 to $det(A(x))$, we have $|det(A(\alpha_1))| = |\Pi_{i \neq j}(\alpha_i - \alpha_j)|$ and the lemma is proved by induction.