POW 2024-10: Supremum

수리과학과 20학번 김준홍

1. $\sum_{n=1}^{N} \frac{1}{\sqrt{n}} \left(\sum_{i=n}^{N} x_i^2 \right)^{1/2} \Big/ \sum_{i=1}^{N} x_i$ has maximum value $\sum_{n=1}^{N} \frac{1}{N} \frac{\sqrt{N+1-n}}{\sqrt{n}}$ when $x_1 = \cdots = x_N$.

Proof) ■

Note that $(x_1, \cdots, x_n, x_{n+1})$ with $x_1 = \cdots = x_{n+1}$ gives larger value than $(x_1, \cdots, x_n, 0)$, so $\sum_{n=1}^{N} \frac{1}{N} \frac{\sqrt{N+1-n}}{\sqrt{n}}$ increases as N gets larger.

2. $A := \sup \left[\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(\sum_{i=n}^{\infty} x_i^2 \right)^{1/2} \Big/ \sum_{i=1}^{\infty} x_i \right] = \sup_{N \in \mathbb{N}} \left[\sum_{n=1}^{N} \frac{1}{N} \frac{\sqrt{N+1-n}}{\sqrt{n}} \right] =: B.$

Proof) Note that $f(\mathbf{x}) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(\sum_{i=n}^{\infty} x_i^2 \right)^{1/2} \Big/ \sum_{i=1}^{\infty} x_i$ is continuous function for $x_i > 0$, so there exists $\delta > 0$ such that $|\mathbf{x} - \mathbf{y}| < \delta$ implies $|f(\mathbf{x}) - f(\mathbf{y})| < \epsilon$.

Fix some sequence $\{x_i\}$ and $\epsilon > 0$. Since $\sum_{i=1}^{\infty} x_i < \infty$, we can choose N such that $\sum_{i=N+1}^{\infty} x_i < \delta$.

Then for $\mathbf{y}$ such that $y_i = x_i$ for $i = 1, \cdots, N$ and $y_i = 0$ for $i > N$,

$|\mathbf{x} - \mathbf{y}| = \sqrt{\sum_{i=N+1}^{\infty} x_i^2} \leq \sum_{i=N+1}^{\infty} x_i < \delta$, so $|f(\mathbf{x}) - f(\mathbf{y})| < \epsilon$.

As ϵ is arbitrary, we can construct sequence $\mathbf{y}^{(j)}$ such that $f(\mathbf{y}^{(\mathbf{j})})$ converges to $f(\mathbf{x})$. Therefore, $f(\mathbf{x}) \leq \sup_j [f(\mathbf{y}^{(\mathbf{j})})] \leq B$. As this holds for arbitrary $\mathbf{x}$, take supremum and we get $A \leq B$.

For opposite, for arbitrary $\mathbf{y} = (a, \cdots, a, 0, \cdots, 0, \cdots)$ we can also construct positive sequence $\mathbf{x}^{(j)}$ such that $f(\mathbf{x}^{(\mathbf{j})})$ converges to $f(\mathbf{y})$. So $f(\mathbf{y}) \leq \sup_j [f(\mathbf{x}^{(\mathbf{j})})] \leq A$.

As this holds for arbitrary $\mathbf{y}$, take supremum and we get $B \leq A$. ■

3. $\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{\sqrt{N+1-n}}{\sqrt{n}} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{\sqrt{1-n/(N+1)}}{\sqrt{n/(N+1)}} = \lim_{N \to \infty} \frac{1}{N+1} \sum_{n=1}^{N+1} \frac{\sqrt{1-n/(N+1)}}{\sqrt{n/(N+1)}}$

$= \int_0^1 \frac{\sqrt{1-x}}{\sqrt{x}} \mathrm{dx}$. Let $\sin(t) = \sqrt{x}$ and $\cos(t) = \sqrt{1-x}$ $(0 \leq t \leq \pi/2)$.

Then $\cos(t)\mathrm{dt} = \frac{1}{2\sqrt{x}}\mathrm{dx} = \frac{1}{2\sin(t)}\mathrm{dx}$, so $\mathrm{dx} = 2\cos(t)\sin(t)\mathrm{dt}$.

So $\displaystyle\int_0^1 \frac{\sqrt{1-x}}{\sqrt{x}}\mathrm{dx} = \int_0^{\pi/2} \frac{\cos(t)}{\sin(t)} 2\cos(t)\sin(t)\mathrm{dt} = \int_0^{\pi/2} 2\cos^2(t)\mathrm{dt} = \frac{\pi}{2}$.

By note in **1** and **2**, sup $\displaystyle\left[\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(\sum_{i=n}^{\infty} x_i^2\right)^{1/2} \Big/ \sum_{i=1}^{\infty} x_i\right] = \frac{\pi}{2}.$ ■