# POW 2024-09

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**(Problem)** Find all positive numbers $a_1, a_2, \cdots, a_5$ such that $a_1^{\frac{1}{n}} + a_2^{\frac{1}{n}} + \cdots + a_5^{\frac{1}{n}}$ is integer for every integer $n \geq 1$.

**(Solution)**

Let $a_1, a_2, \cdots, a_5$ be positive numbers such that $a_1^{\frac{1}{n}} + a_2^{\frac{1}{n}} + \cdots + a_5^{\frac{1}{n}}$ is integer for every $n \in N$.

Then, for each $i \in \{1,2,3,4,5\}$, there exists a natural number $N_i$ such that $\frac{4}{5} < a_i^{\frac{1}{n}} < \frac{6}{5}$ for all $n \geq N_i$. $\left(\because \lim_{n\to\infty} a_i^{\frac{1}{n}} = 1.\right)$

Define $M := \max\{N_1, N_2, \cdots, N_5\}$.

If $n \geq M$, then $4 < a_1^{\frac{1}{n}} + a_2^{\frac{1}{n}} + \cdots + a_5^{\frac{1}{n}} < 6$ implying that $a_1^{\frac{1}{n}} + a_2^{\frac{1}{n}} + \cdots + a_5^{\frac{1}{n}} = 5$.

Therefore, $a_1^{\frac{1}{M}} + a_2^{\frac{1}{M}} + \cdots + a_5^{\frac{1}{M}} = a_1^{\frac{1}{2M}} + a_2^{\frac{1}{2M}} + \cdots + a_5^{\frac{1}{2M}} = 5$.

$$\left(a_1^{\frac{1}{M}} + a_2^{\frac{1}{M}} + \cdots + a_5^{\frac{1}{M}}\right)(1^2+1^2+1^2+1^2+1^2) \geq \left(a_1^{\frac{1}{2M}} + a_2^{\frac{1}{2M}} + \cdots + a_5^{\frac{1}{2M}}\right)^2 \quad \cdots (*)$$

Note that, by Cauch-Schwarz inequality, (*) holds and equality holds iff $a_1^{\frac{1}{2M}} = a_2^{\frac{1}{2M}} = \cdots = a_5^{\frac{1}{2M}}$.

Since, $\left(a_1^{\frac{1}{M}} + a_2^{\frac{1}{M}} + \cdots + a_5^{\frac{1}{M}}\right)(1^2+1^2+1^2+1^2+1^2) = \left(a_1^{\frac{1}{2M}} + a_2^{\frac{1}{2M}} + \cdots + a_5^{\frac{1}{2M}}\right)^2 = 25$,

$a_1^{\frac{1}{2M}} = a_2^{\frac{1}{2M}} = \cdots = a_5^{\frac{1}{2M}}$ and $a_1 = a_2 = \cdots = a_5 = 1$.

$\therefore$ The only positive integers $a_1 \sim a_5$ such that $a_1^{\frac{1}{n}} + a_2^{\frac{1}{n}} + \cdots + a_5^{\frac{1}{n}}$ is integer for every $n \in N$ are $a_1 = a_2 = \cdots = a_5 = 1$.