08 Let A be a 16×16 matrix whose entries are either 1 or -1. What is the maximum value of the determinant of A?

Solution. Let $A = [a_{ij}] \in \{\pm 1\}^{n\times n}$ and $B = [b_{ij}] = A^T A$. Note that

$$\operatorname{tr} B = \sum_{i=1}^{n} b_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 = n^2.$$

Since B is symmetric, by the spectral theorem, it is orthogonally diagonalizable. There exists some orthogonal matrix Q such that

$$Q^T B Q = \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

where λ_j's are the eigenvalues of B. By the AM–GM inequality, we have

$$\det B = \det \Lambda = \prod_{j=1}^{n} \lambda_j \le \left(\frac{1}{n}\sum_{j=1}^{n}\lambda_j\right)^n = \left(\frac{\operatorname{tr} B}{n}\right)^n = n^n.$$

Hence, we obtain a bound $|\det A| = \sqrt{\det B} \le n^{n/2}$. The equality holds when

$$\Lambda = nI_n = Q^T B Q = Q^T A^T A Q = (AQ)^T (AQ).$$

We can construct a sequence of $H_n \in \{\pm 1\}^{n\times n}$ such that $H_n^T H_n = nI_n$ as follows. Let $H_1 = \begin{bmatrix} 1 \end{bmatrix}$ and define

$$H_{2^{k+1}} = \begin{bmatrix} H_{2^k} & H_{2^k} \\ H_{2^k} & -H_{2^k} \end{bmatrix}.$$

For example,

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Furthermore, $\det H_{2^{k+1}} = \det\left(-2H_{2^k}^2\right) = (-2)^{2^k}\left(\det H_{2^k}\right)^2 > 0$ if $k > 0$. Therefore, there is a matrix $H_{16} \in \{\pm 1\}^{16\times 16}$ such that $\det H_{16} = 16^8 = 2^{32}$. □

Remark. This problem is called *Hadamard's maximal determinant problem*, and H_n is known as a *Hadamard matrix* of order n. The method of constructing a sequence $\{H_{2^k}\}$ of Hadamard matrices was first introduced by James Joseph Sylvester in 1867.