Let $A$ be a $16 \times 16$ matrix whose entries are either $1$ or $-1$. What is the maximum value of the determinant of $A$?

**Solution.** Let $A = [a_{ij}] \in \{\pm 1\}^{n \times n}$ and $B = [b_{ij}] = A^T A$. Note that

$$\text{tr } B = \sum_{i=1}^{n} b_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 = n^2.$$  

Since $B$ is symmetric, by the spectral theorem, it is orthogonally diagonalizable. There exists some orthogonal matrix $Q$ such that

$$Q^T B Q = \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

where $\lambda_j$'s are the eigenvalues of $B$. By the AM–GM inequality, we have

$$\det B = \det \Lambda = \prod_{j=1}^{n} \lambda_j \leq \left( \frac{1}{n} \sum_{j=1}^{n} \lambda_j \right)^n = \left( \frac{\text{tr } B}{n} \right)^n = n^n.$$  

Hence, we obtain a bound $|\det A| = \sqrt[n]{\det B} \leq n^{n/2}$. The equality holds when

$$\Lambda = nI_n = Q^T B Q = Q^T A^T A Q = (AQ)^T (AQ).$$

We can construct a sequence of $H_n \in \{\pm 1\}^{n \times n}$ such that $H_n^T H_n = nI_n$ as follows. Let $H_1 = \begin{bmatrix} 1 \end{bmatrix}$ and define

$$H_{2k+1} = \begin{bmatrix} H_{2k} & H_{2k} \\ H_{2k} & -H_{2k} \end{bmatrix}.$$  

For example,

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$  

Furthermore, $\det H_{2k+1} = \det (-2H_{2k}^2) = (-2)^{2k} \left( \det H_{2k} \right)^2 > 0$ if $k > 0$. Therefore, there is a matrix $H_{16} \in \{\pm 1\}^{16 \times 16}$ such that $\det H_{16} = 16^8 = 2^{32}$.

**Remark.** This problem is called Hadamard’s maximal determinant problem, and $H_n$ is known as a Hadamard matrix of order $n$. The method of constructing a sequence $\{H_{2k}\}$ of Hadamard matrices was first introduced by James Joseph Sylvester in 1867.