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We assume that \( k \leq m \), otherwise the sequence is not well defined. By ignoring the first \( m-k \) terms, without loss of generality we may assume that \( k = m \). Let \( f(x_1, \ldots, x_m) = x_1^r + x_2^r + \cdots + x_m^r \) so that
\[
a_n = f(a_{n-1}, \ldots, a_{n-m}) = n > m.
\]
Define \( a = m^{\frac{1}{r}} \), and \( g(x) := f(x, x, \ldots, x) = mx^r \). Then it is clear that
\[
\begin{align*}
g(x) &> x \quad \text{if } x < a, \\
g(a) &= a, \\
g(x) &< x \quad \text{if } x > a.
\end{align*}
\]

The function \( g \) also has the following property.

**Proposition 1.** Let \( g^k(x) = g \circ \cdots \circ g(x) \). Then for any \( k = 0, 1, \ldots \), it holds that \( g^k(x) = m^{\sum_{i=0}^{k-1} r^i} x^{r^k} \). In particular, \( \lim_{k \to \infty} g^k(x) = a \).

**Proof.** For the first part of the statement, we use induction on \( k \). For the case \( k = 0 \) there is nothing to show. Now suppose that \( g^k(x) = m^{\sum_{i=0}^{k-1} r^i} x^{r^k} \) holds for some \( k \), then
\[
g^{k+1}(x) = g\left(g^k(x)\right) = m^{\sum_{i=0}^{k} r^i} x^{r^{k+1}},
\]
so we are done.

For the second part of the statement, notice that because \( 0 < r < 1 \) we have \( r^k \to 0 \) and \( \sum_{i=0}^{\infty} r^i = \frac{1}{1-r} \). The conclusion is then immediate.

In order to find the answer to the given problem, let us first examine the special case where all initial values \( a_1, \ldots, a_m \) are equal.

**Lemma 2.** Suppose that \( a_1 = a_2 = \cdots = a_m = \gamma \) for some \( \gamma > 0 \). Then \( \lim_{n \to \infty} a_n = a \).

**Proof.** Suppose that \( \gamma = a \), then by \([\square]\) the sequence becomes a constance sequence \( a_n = a \).

Suppose that \( \gamma > a \). Note that, by definition, \( f \) is increasing on each argument, and also invariant under the permutation of the arguments.

**Claim 1.** \( \{a_n\}_{n \geq 1} \) is a decreasing sequence.

We use strong induction on \( n \). By assumption we have \( a_1 = \cdots = a_m \), and
\[
a_{m+1} = f(a_m, a_m, \ldots, a_m) = f(\gamma, \ldots, \gamma) = g(\gamma) < \gamma = a_m
\]
where the inequality follows from \([\square]\). Now for some \( n > m \), suppose that \( a_1 \geq a_2 \geq \cdots \geq a_n \). Then
\[
a_{n+1} = f(a_n, a_{n-1}, \ldots, a_{n-m+1})
\leq f(a_{n-m}, a_{n-1}, \ldots, a_{n-m+1})
= f(a_{n-1}, \ldots, a_{n-m+1}, a_{n-m}) = a_n
\]
where the second line follows from that \( a_{n-m} \geq a_n \), and the third line follows from the invariance of \( f \) under the permutation of its arguments. This completes the induction step.
Claim 2  \( \{a_n\}_{n \geq 1} \) is bounded below by \( \alpha \).

We use strong induction on \( n \). By assumption we have \( a_1 = \cdots = a_m = \gamma > \alpha \). Now for some \( n \geq m \), suppose that \( a_k > \alpha \) for all \( k \leq n \). Then

\[
a_{n+1} = f(a_n, a_{n-1}, \ldots, a_{n-m+1}) \\
\leq f(a_n, a_n, \ldots, a_n) = g(a_n) < a_n
\]

where from the first line to the second line we used the fact that \( \{a_n\}_{n \geq 1} \) is a decreasing sequence, and in the last inequality we used (*)). This completes the induction step.

From Claims 1 and 2, we know that \( \{a_n\}_{n \geq 1} \) is a convergent sequence. Now, notice that for any \( k \geq 1 \), we have

\[
a_{km+1} = f(a_{km}, a_{km-1}, \ldots, a_{km-m+1}) \\
\leq f(a_{km-m+1}, a_{km-m+1}, \ldots, a_{km-m+1}) \\
= g(a_{k-1}m+1)
\]

and hence \( a_{km+1} \leq g^k(a_1) \). By Proposition 1, we have \( g^k(a_1) \to \alpha \) as \( k \to \infty \), so by the sandwich theorem, \( \lim_{k \to \infty} g^k(a_1) = \alpha \). It follows that \( \lim_{n \to \infty} a_n = \alpha \).

Finally, suppose that \( \gamma < \alpha \). Then in fact, we can show that \( \{a_n\}_{n \geq 1} \) is an increasing sequence bounded above by \( \alpha \), by using the exact same logic used in Claims 1 and 2 but only all the inequalities reversed. Hence, the sequence \( \{a_n\}_{n \geq 1} \) is convergent, and moreover, for any \( k \geq 1 \) we have

\[
a_{km+1} = f(a_{km}, a_{km-1}, \ldots, a_{km-m+1}) \\
\geq f(a_{km-m+1}, a_{km-m+1}, \ldots, a_{km-m+1}) \\
= g(a_{k-1}m+1)
\]

where the second line follows from that \( \{a_n\}_{n \geq 1} \) is increasing. Consequently, \( a_{km+1} \geq g^k(a_1) \). Then again, by Proposition 1, \( \lim_{k \to \infty} g^k(a_1) = \alpha \), so by the sandwich theorem, \( \lim_{k \to \infty} a_{km+1} = \alpha \). It follows that \( \lim_{n \to \infty} a_n = \alpha \) in this case also, so we are done.

Now we can answer to the given problem. Define \( \mu := \min\{a_1, \ldots, a_m\} \), \( M := \max\{a_1, \ldots, a_m\} \), and let us consider two auxiliary sequences \( \{z_n\}_{n \geq 1} \) and \( \{b_n\}_{n \geq 1} \) defined as

\[
z_n = \begin{cases} 
\mu & \text{if } n \leq m, \\
f(z_{n-1}, \ldots, z_{n-m}) & \text{if } n > m,
\end{cases}
\quad b_n = \begin{cases} 
M & \text{if } n \leq m, \\
f(b_{n-1}, \ldots, b_{n-m}) & \text{if } n > m.
\end{cases}
\]

Let us use strong induction on \( n \) to show that \( z_n \leq a_n \leq b_n \). By definition, for all \( k = 1, \ldots, m \) we have \( z_k = \mu \leq a_k \leq M = b_k \). Now suppose that, for some \( n \geq m \), it holds that \( z_k \leq a_k \leq b_k \) for all \( k = 1, \ldots, n \). Then we get

\[
z_{n+1} = z_n^r + z_{n-1}^r + \cdots + z_{n-m+1}^r \\
\leq a_n^r + a_{n-1}^r + \cdots + a_{n-m+1}^r = a_{n+1} \\
\leq b_n^r + b_{n-1}^r + \cdots + b_{n-m+1}^r = b_{n+1},
\]

so we are done.

Meanwhile, by Lemma 2, we have both \( \lim_{n \to \infty} z_n = \alpha \) and \( \lim_{n \to \infty} b_n = \alpha \). Therefore, by the sandwich theorem, we conclude that \( \lim_{n \to \infty} a_n = \alpha \).