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We assume that $k \le m$, otherwise the sequence is not well defined. By ignoring the first $m-k$ terms, without loss of generality we may assume that $k = m$. Let $f(x_1, \dots, x_m) = x_1^r + x_2^r + \cdots + x_m^r$ so that

$$a_n = f(a_{n-1}, \dots, a_{n-m}), \quad n > m.$$

Define $\alpha = m^{\frac{1}{1-r}}$, and $g(x) := f(x, x, \dots, x) = mx^r$. Then it is clear that

$$\begin{aligned} &g(x) > x \quad \text{if } x < \alpha, \\ &g(\alpha) = \alpha, \\ &g(x) < x \quad \text{if } x > \alpha. \end{aligned} \tag{*}$$

The function g also has the following property.

Proposition 1. *Let* $g^k(x) = \underbrace{g \circ \cdots \circ g}_{k \text{ times}}(x)$. *Then for any* $k = 0, 1, \dots$, *it holds that* $g^k(x) = m^{\sum_{i=0}^{k-1} r^i} x^{r^k}$. *In particular,* $\lim_{k\to\infty} g^k(x) = \alpha$.

Proof. For the first part of the statement, we use induction on k. For the case $k = 0$ there is nothing to show. Now suppose that $g^k(x) = m^{\sum_{i=0}^{k-1} r^i} x^{r^k}$ holds for some k, then

$$g^{k+1}(x) = g\left(g^k(x)\right) = m\left(m^{\sum_{i=0}^{k-1} r^i} x^{r^k}\right)^r = m^{\sum_{i=0}^{k} r^i} x^{r^{k+1}},$$

so we are done.

For the second part of the statement, notice that because $0 < r < 1$ we have $r^k \to 0$ and $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$. The conclusion is then immediate. □

In order to find the answer to the given problem, let us first examine the special case where all initial values $a_1, \dots, a_m$ are equal.

Lemma 2. *Suppose that* $a_1 = a_2 = \cdots = a_m = \gamma$ *for some* $\gamma > 0$. *Then* $\lim_{n\to\infty} a_n = \alpha$.

Proof. Suppose that $\gamma = \alpha$, then by (*) the sequence becomes a constance sequence $a_n = \alpha$.

Suppose that $\gamma > \alpha$. Note that, by definition, f is increasing on each argument, and also invariant under the permutation of the arguments.

Claim 1 $\{a_n\}_{n\ge1}$ is a decreasing sequence.

We use strong induction on n. By assumption we have $a_1 = \cdots = a_m$, and

$$a_{m+1} = f(a_m, \dots, a_1) = f(\gamma, \dots, \gamma) = g(\gamma) < \gamma = a_m$$

where the inequality follows from (*). Now for some $n > m$, suppose that $a_1 \ge a_2 \ge \cdots \ge a_n$. Then

$$\begin{aligned} a_{n+1} &= f(a_n, a_{n-1}, \dots, a_{n-m+1}) \\ &\le f(a_{n-m}, a_{n-1}, \dots, a_{n-m+1}) \\ &= f(a_{n-1}, \dots, a_{n-m+1}, a_{n-m}) = a_n \end{aligned}$$

where the second line follows from that $a_{n-m} \ge a_n$, and the third line follows from the invariance of f under the permutation of its arguments. This completes the induction step.

Claim 2 $\{a_n\}_{n\geq 1}$ is bounded below by α.

We use strong induction on n. By assumption we have $a_1 = \cdots = a_m = \gamma > \alpha$. Now for some $n \geq m$, suppose that $a_k > \alpha$ for all $k \leq n$. Then

$$\begin{aligned} a_{n+1} &= f(a_n, a_{n-1}, \ldots, a_{n-m+1}) \\ &\leq f(a_n, a_n, \ldots, a_n) = g(a_n) < a_n \end{aligned}$$

where from the first line to the second line we used the fact that $\{a_n\}_{n\geq 1}$ is a decreasing sequence, and in the last inequality we used (*). This completes the induction step.

From Claims 1 and 2, we know that $\{a_n\}_{n\geq 1}$ is a convergent sequence. Now, notice that for any $k \geq 1$, we have

$$\begin{aligned} a_{km+1} &= f(a_{km}, a_{km-1}, \ldots, a_{km-m+1}) \\ &\leq f(a_{km-m+1}, a_{km-m+1}, \ldots, a_{km-m+1}) \\ &= g(a_{(k-1)m+1}) \end{aligned}$$

and hence $a_{km+1} \leq g^k(a_1)$. By Proposition 1, we have $g^k(a_1) \to \alpha$ as $k \to \infty$, so by the sandwich theorem, $\lim_{k\to\infty} a_{km+1} = \alpha$. It follows that $\lim_{n\to\infty} a_n = \alpha$.

Finally, suppose that $\gamma < \alpha$. Then in fact, we can show that $\{a_n\}_{n\geq 1}$ is an increasing sequence bounded above by α, by using the exact same logic used in Claims 1 and 2 but only all the inequalities reversed. Hence, the sequence $\{a_n\}_{n\geq 1}$ is convergent, and moreover, for any $k \geq 1$ we have

$$\begin{aligned} a_{km+1} &= f(a_{km}, a_{km-1}, \ldots, a_{km-m+1}) \\ &\geq f(a_{km-m+1}, a_{km-m+1}, \ldots, a_{km-m+1}) \\ &= g(a_{(k-1)m+1}) \end{aligned}$$

where the second line follows from that $\{a_n\}_{n\geq 1}$ is increasing. Consequently, $a_{km+1} \geq g^k(a_1)$. Then again, by Proposition 1, $\lim_{k\to\infty} g^k(a_1) = \alpha$, so by the sandwich theorem, $\lim_{k\to\infty} a_{km+1} = \alpha$. It follows that $\lim_{n\to\infty} a_n = \alpha$ in this case also, so we are done. □

Now we can answer to the given problem. Define $\mu := \min\{a_1, \ldots, a_m\}$, $M := \max\{a_1, \ldots, a_m\}$, and let us consider two auxiliary sequences $\{z_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ defined as

$$z_n = \begin{cases} \mu & \text{if } n \leq m, \\ f(z_{n-1}, \ldots, z_{n-m}) & \text{if } n > m, \end{cases} \quad \text{and} \quad b_n = \begin{cases} M & \text{if } n \leq m, \\ f(b_{n-1}, \ldots, b_{n-m}) & \text{if } n > m. \end{cases}$$

Let us use strong induction on n to show that $z_n \leq a_n \leq b_n$. By definition, for all $k = 1, \ldots, m$ we have $z_k = \mu \leq a_k \leq M = b_k$. Now suppose that, for some $n \geq m$, it holds that $z_k \leq a_k \leq b_k$ for all $k = 1, \ldots, n$. Then we get

$$\begin{aligned} z_{n+1} &= z_n^r + z_{n-1}^r + \cdots + z_{n-m+1}^r \\ &\leq a_n^r + a_{n-1}^r + \cdots + a_{n-m+1}^r = a_{n+1} \\ &\leq b_n^r + b_{n-1}^r + \cdots + b_{n-m+1}^r = b_{n+1}, \end{aligned}$$

so we are done.

Meanwhile, by Lemma 2, we have both $\lim_{n\to\infty} z_n = \alpha$ and $\lim_{n\to\infty} b_n = \alpha$. Therefore, by the sandwich theorem, we conclude that $\lim_{n\to\infty} a_n = \alpha$.