POW 2024-05: Knotennullstelle

수리과학과 20학번 김준홍

Proposition) For nth cyclotomic polynomial $\Phi_n(x)$,

(0) $n = \prod_{1 \neq d | n} \Phi_d(1)$

(1) $\Phi_n(1) = p$ if $n = p^r$ for prime p and $r \geq 1$

(2) $\Phi_n(1) = 1$ if $n = p_1^{r_1} \cdots p_m^{r_m}$ for distinct prime p_i and $r_i \geq 1$ for $1 \leq i \leq m$.

Proof)

(0) Note that $x^n - 1 = \prod_{d|n} \Phi_d(x)$. So $\frac{x^n - 1}{\Phi_1(x)} = \frac{x^n - 1}{x - 1} = x^{n-1} + \cdots + x + 1 = \prod_{1 \neq d|n} \Phi_d(x)$.
Put $x = 1$, then $n = \prod_{1 \neq d|n} \Phi_d(1)$.

(1) Use strong induction in r. For $r = 1$, $p = \prod_{1 \neq d|n} \Phi_d(1) = \Phi_d(p)$.
Assume that it holds for $r = 1, \cdots, k$. Then $p^{k+1} = \Phi_p(1) \cdots \Phi_{p^k}(1)\Phi_{p^{k+1}}(1) = p^k \Phi_{p^{k+1}}(1)$ by induction, so $\Phi_{p^{k+1}}(1) = p$.

(2) Use strong induction in $r = r_1 + \cdots + r_m$. For $r = 2$, $p_1 p_2 = \Phi_{p_1}(1)\Phi_{p_2}(1)\Phi_{p_1 p_2}(1)$, so $\Phi_{p_1 p_2}(1) = 1$. Assume that it holds for $r = 2, \cdots, k$. Then for $r = k+1$, $p_1^{r_1} \cdots p_m^{r_m} = \prod_{1 \neq d|n} \Phi_d(1)$.
Since $\Phi_{p_i^a}(1) = p_i$ for all $1 \leq a \leq r_i$ and $1 \leq i \leq m$ by (1), and $\Phi_d(1) = 1$ for every composite d except n by induction, $\Phi_n(1) = 1$.

Claim) Collection of all Knotennullstelle numbers is **not** a discrete subset of $\mathbb{C}$.

Proof) By proposition, we can know that $z = e^{2\pi i m/n}$ with $\gcd(m, n) = 1$ and n composite is Knotennullstelle, as $\Phi_n(z) = 0$ and $\Phi_n(1) = 1$. If converging Knotennullstelle sequence $\{z_n\}$ exists such that limit is also Knotennullstelle, then claim holds. ($\because$ for any neighborhood of $\lim_{n\to\infty} z_n$, there exists some z_k in the neighborhood) One example can be $z_n = e^{2\pi i c_n}$ for $c_n = \frac{1}{5}\sum_{i=1}^{n} \frac{1}{3^i}$. As c_n has denominator $5 \cdot 3^n$, z_n is Knotennullstelle. Also, $\lim_{n\to\infty} c_n = \frac{1}{10}$, so $\lim_{n\to\infty} z_n$ is Knotennullstelle. Therefore, claim holds. ■