POW 2024-05: Knotennullstelle

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**Proposition**) For $n$th cyclotomic polynomial $\Phi_n(x)$,

(0) $n = \prod_{1 \neq d|n} \Phi_d(1)$

(1) $\Phi_n(1) = p$ if $n = p^r$ for prime $p$ and $r \geq 1$

(2) $\Phi_n(1) = 1$ if $n = p_1^{r_1} \cdots p_m^{r_m}$ for distinct prime $p_i$ and $r_i \geq 1$ for $1 \leq i \leq m$.

**Proof**

(0) Note that $x^n - 1 = \prod_{d|n} \Phi_d(x)$. So $\frac{x^n - 1}{\Phi_1(x)} = \frac{x^n - 1}{x - 1} = x^{n-1} + \cdots + x + 1 = \prod_{1 \neq d|n} \Phi_d(x)$.

Put $x = 1$, then $n = \prod_{1 \neq d|n} \Phi_d(1)$.

(1) Use strong induction in $r$. For $r = 1$, $p = \prod_{1 \neq d|n} \Phi_d(1) = \Phi_d(p)$.

Assume that it holds for $r = 1, \cdots, k$. Then $p^{k+1} = \Phi_1(1) \cdots \Phi_k(1) \Phi_{p^{k+1}}(1) = p^k \Phi_{p^{k+1}}(1)$ by induction, so $\Phi_{p^{k+1}}(1) = p$.

(2) Use strong induction in $r = r_1 + \cdots + r_m$. For $r = 2$, $p_1p_2 = \Phi_{p_1}(1) \Phi_{p_2}(1) \Phi_{p_1p_2}(1)$, so $\Phi_{p_1p_2}(1) = 1$. Assume that it holds for $r = 2, \cdots, k$. Then for $r = k+1$, $p_1^{r_1} \cdots p_m^{r_m} = \prod_{1 \neq d|n} \Phi_d(1)$.

Since $\Phi_{p_i}(1) = p_i$ for all $1 \leq a \leq r_i$ and $1 \leq i \leq m$ by (1), and $\Phi_d(1) = 1$ for every composite $d$ except $n$ by induction, $\Phi_n(1) = 1$.

**Claim**) Collection of all Knotennullstelle numbers is not a discrete subset of $\mathbb{C}$.

**Proof**) By proposition, we can know that $z = e^{2\pi im/n}$ with $\gcd(m, n) = 1$ and $n$ composite is Knotennullstelle, as $\Phi_n(z) = 0$ and $\Phi_n(1) = 1$. If converging Knotennullstelle sequence \( \{z_n\} \) exists such that limit is also Knotennullstelle, then claim holds. (\( \because \) for any neighborhood of \( \lim_{n \to \infty} z_n \), there exists some \( z_k \) in the neighborhood) One example can be $z_n = e^{2\pi icn}$ for \( c_n = \frac{1}{5} \sum_{i=1}^{n} \frac{1}{3^i} \). As $c_n$ has denominator $5 \cdot 3^n$, $z_n$ is Knotennullstelle. Also, $\lim_{n \to \infty} c_n = \frac{1}{10}$, so $\lim_{n \to \infty} z_n$ is Knotennullstelle. Therefore, claim holds. ■