POW 2024-04: Real random variable

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Theorem) (Cauchy-Schwarz inequality for random variables)

For real random variables X, Y, $(E[XY])^2 \leq E[X^2]E[Y^2]$.

Proof) $0 \leq E[(tX+Y)^2] = E[t^2X^2 + 2tXY + Y^2] = E[X^2]t^2 + 2tE[XY] + E[Y^2]$ for all $t \in \mathbb{R}$.

So $D/4 = (E[XY])^2 - E[X^2]E[Y^2] \leq 0$. ■

($\Rightarrow$) Put $X = Z^2 - 1, Y = Z$ in above theorem. Then $(E[(Z^2-1)Z])^2 \leq E[(Z^2-1)^2]E[Z^2]$,

$x^2 = (E[Z^3] - E[Z])^2 = (E[(Z^2-1)Z])^2 \leq E[(Z^2-1)]E[Z^2] = E[Z^4 - 2Z^2 + 1]E[Z^2] = y - 1$.

($\Leftarrow$) Construct Z as $P(Z=0) = c, P(Z=a) = d, P(Z=b) = e$.

Then we have 5 conditions hold for given x, y, and 3 identites:

$$d + e = 1 - c$$

$$ad + be = 0 \qquad a^2d + b^2e = (a+b)(ad+be) - ab(d+e)$$

$$a^2d + b^2e = 1 \qquad a^3d + b^3e = (a+b)(a^2d+b^2e) - ab(ad+be)$$

$$a^3d + b^3e = x \qquad a^4d + b^4e = (a+b)(a^3d+b^3e) - ab(a^2d+b^2e)$$

$$a^4d + b^4e = y$$

By these equations, we get $1 = -ab(1-c), x = a+b, y = (a+b)x - ab$.

Then $a + b = x, ab = x^2 - y$, so $a, b = \dfrac{x \pm \sqrt{4y - 3x^2}}{2}$.

Note that $4y - 3x^2 \geq 4(x^2+1) - 3x^2 = x^2 + 4 > 0$, so real roots a, b exists.

Also, $c = 1 + \dfrac{1}{ab}$. As $ab = x^2 - y \leq -1$, $c \geq 0$.

For $d + e = 1 - c, ad + be = 0$, we get $d = \dfrac{1}{b(b-a)}, e = \dfrac{1}{a(a-b)}$.

As $d = \dfrac{1}{b^2 - ab} > 0$, $e = \dfrac{1}{a^2 - ab} > 0$ and $c + d + e = 1$, Z is actually random variable.

Therefore, Z satisfies given condition.