# POW 2024-03: Roots of complex derivative

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If $a = b$, then line segment is just $a$, which means $\mathrm{Re}(P'(a)) = 0$. But in this case, $P(z) = z^3 - 4z$ is counterexample. $P(0) = 0$ but $\mathrm{Re}(P'(0)) = -4$, so statement does not hold. So let $a \neq b$.

**Lemma**) $\frac{\partial}{\partial u}[\mathrm{Re}(P(z))] = \mathrm{Re}(P'(z))$ for complex polynomial $P(z)$, and $z = u + iv$.

**Proof**) It is sufficient to show when $P(z) = cz^n$. If $n = 0$, then both side are 0, so it holds. Note that $\frac{\partial z}{\partial u} = \frac{\partial \overline{z}}{\partial u} = 1$. Otherwise, $\frac{\partial}{\partial u}[\mathrm{Re}(P(z))] = \frac{\partial}{\partial u}(\frac{cz^n + \overline{cz^n}}{2}) = \frac{\partial}{\partial u}(\frac{cz^n + \overline{c}\overline{z}^n}{2}) = \frac{cnz^{n-1} + \overline{c}n\overline{z}^{n-1}}{2} = \frac{cnz^{n-1} + \overline{cnz^{n-1}}}{2} = \mathrm{Re}(P'(z))$. ■

For line segment joining $a$ and $b$, we can rotate whole plane so that segment is parallel to real axis. In other words, $Q(z) = P(e^{i\theta}z)$ has zero at $a' = ae^{-i\theta}$ and $b' = be^{-i\theta}$, and $\mathrm{I}m(a') = \mathrm{I}m(b')$.

Consider $f(u) = \mathrm{Re}(e^{-i\theta}Q(z))$ for $z = u + \mathrm{I}m(a')$. We can consider this as single-valued function, as imaginary part is constant in line segment joining $a'$ and $b'$. Note that $f(\mathrm{Re}(a')) = f(\mathrm{Re}(b')) = 0$, and $f(u)$ is real function. By Rolle's theorem, there exists some $u'$ between $\mathrm{Re}(a')$ and $\mathrm{Re}(b')$ such that $f'(u') = \frac{\partial}{\partial u}[\mathrm{Re}(e^{-i\theta}Q(w))] = \mathrm{Re}(e^{-i\theta}Q'(w)) = 0$ for $w = u' + i\mathrm{I}m(a')$. Since $Q(z) = P(e^{i\theta}z)$ implies $Q'(z) = e^{i\theta}P'(e^{i\theta}z)$, $\mathrm{Re}(P'(e^{i\theta}w)) = \mathrm{Re}(e^{-i\theta}Q'(w)) = 0$. $w$ belongs to segment of $a' = ae^{-i\theta}$ and $b' = be^{-i\theta}$, so $e^{i\theta}w$ belongs to segment of $a$ and $b$, which is the desired result.