Well-mixed permutations

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Problem. A permutation $\phi\colon \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ is called a *well-mixed* if $\phi(\{1,2,\ldots,k\}) \neq \{1,2,\ldots,k\}$ for each $k < n$. What is the number of well-mixed permutations of $\{1,2,\ldots,15\}$?

Solution.

Notation. $[a,b]$ is a set of integers n satisfying $a \le n \le b$. $[1,n]$ is sometimes simply denoted as $[n]$. The set of all permutations of $[n]$ is denoted as S_n. Let ω_n be the number of well-mixed permutations of $[n]$. For the formal power series $F(x) = \sum_{n\ge 0} f_n x^n$, define $[x^n] F(x)$ as f_n.

Definition. For $\sigma_1 \in S_{n_1}, \ldots, \sigma_r \in S_{n_r}$, the *merged permutation* $\pi = [\sigma_1, \ldots, \sigma_r] \in S_{n_1+\cdots+n_r}$ of $\sigma_1, \ldots, \sigma_r$ is given by

$$\pi(i) = n_1 + \cdots + n_{j-1} + \sigma_j(i - n_1 - \cdots - n_{j-1})$$

where $j \in [r]$ is the smallest number such that $n_1 + \cdots + n_j \ge i$. Empty sums are treated as 0. The partition of $[n_1 + \cdots + n_r]$ in the disjoint intervals $I_i = \left[\sum_{j=1}^{i-1} n_j + 1, \sum_{j=1}^{i} n_j\right]$ is called as the *partition corresponding to* $[\sigma_1, \ldots, \sigma_r]$.

Lemma. Every permutation can be uniquely expressed as a merged permutation of well-mixed permutations.

Proof. We prove this by induction on n in $\sigma \in S_n$. Base case is trivial. For $n > 1$, let $m \in [n]$ be the smallest number such that $\sigma([m]) = [m]$. If $m = n$, σ is itself well-mixed. If $m < n$, we have $\sigma = [\tau, \sigma']$ where τ and σ' are the restrictions of σ to $[m]$ and $[m+1, n]$, respectively. By the induction hypothesis, σ' is a merged permutation of well-mixed permutations, and thus so is σ.

To prove the uniqueness of this expression, assume that

$$\sigma = [\sigma_1, \ldots, \sigma_r] = [\tau_1, \ldots, \tau_s] \in S_n$$

where all σ_i and τ_j are well-mixed. Denote the partitions corresponding to $[\sigma_1, \ldots, \sigma_r]$ and $[\tau_1, \ldots, \tau_s]$ as $I_1, \ldots, I_r$ and $J_1, \ldots, J_s$ respectively. If we let k be the smallest integer such that $I_k \neq J_k$, then $I_k = [\alpha, \beta]$ and $J_k = [\alpha, \gamma]$ with $\beta \neq \gamma$. Without loss of generality, assume $\beta < \gamma$. Then we can see that $\tau_k([\beta - \alpha + 1]) = \sigma_k([\beta - \alpha + 1]) = [\beta - \alpha + 1]$. This contradicts the fact that τ_k is well-mixed. Therefore, there is no such k and the given expression is unique. □

Consider the generating functions $F(x) = \sum_{n\ge 1} \omega_n x^n$ and $G(x) = \sum_{n\ge 1} |S_n| x^n = \sum_{n\ge 1} n! x^n$. According to the above lemma, we can observe the following equation between the two generating functions:

$$1 + G(x) = \sum_{n \geq 0} \{F(x)\}^n \iff F(x) = G(x)[1 - F(x)]$$

It follows that

$$\omega_n = [x^n] F(x) = [x^n] G(x)[1 - F(x)] = n! - \sum_{k=1}^{n-1} (n-k)! \omega_k$$

Now we can get $\omega_{15} = 1123596277863$ through simple calculation. □

Beyond the answer

Define ψ_k as $\omega_k / k!$. Now we can see that $\psi_1 = 1$ and

$$\psi_n = 1 - \sum_{k=1}^{n-1} \psi_k \binom{n}{k}^{-1}$$

Note that $\psi_n \leq 1$ for all $n \in \mathbb{N}$. Then for sufficiently large n, and $m \ll n$, we have

$$\begin{aligned} \sum_{k=1}^{m} \psi_k \binom{n}{k}^{-1} \leq 1 - \psi_n &\leq \sum_{k=1}^{m} \psi_k \binom{n}{k}^{-1} + (n - m + 1) \binom{n}{m+1}^{-1} \\ &\approx \sum_{k=1}^{m} \psi_k \binom{n}{k}^{-1} + \mathcal{O}\left(n^{-m}\right) \end{aligned}$$

Now, if we calculate a few terms of ψ_k directly and do a little bit of labor, we get the following asymptotic expression for ω_n:

$$\omega_n = n! \left[1 - \frac{2}{n} - \frac{1}{(n)_2} - \frac{4}{(n)_3} - \frac{19}{(n)_4} - \frac{110}{(n)_5} + \mathcal{O}\left(n^{-6}\right)\right]$$

where $(n)_k = n(n-1)(n-2)\cdots(n-k+1)$. Some of the n value I calculated and the approximation using the above formula are as follows.

n	ω_n	approximation	relative error
18	5.6617×10^{15}	5.6675×10^{15}	0.1018%
19	1.0836×10^{17}	1.0844×10^{17}	0.0802%
20	2.1811×10^{18}	2.8125×10^{18}	0.0641%
21	4.6067×10^{19}	4.6091×10^{19}	0.0521%
22	1.0187×10^{21}	1.0191×10^{21}	0.0428%

Since the relative error is quite small, it seems that there was no mistake in the asymptotic equation. Phew!

One more thing to note is that when n is very large, most permutations are well-mixed. This matches well with our intuition.