1 Solution

It suffices to prove the following statement, which obviously implies the statement in the problem:

For every positive integer $n$, if $f : \{1, 2, \ldots, n\} \to \mathbb{R}$ is a function such that
(i) $f(1) = 0$, (ii) $f(n) \geq 0$, and (iii) $f(a + b) + f(b + c) + f(c + a) \geq f(a) + f(b) + f(c) + f(a + b + c)$ for all $1 \leq a, b, c \leq n - 2$ with $a + b + c \leq n$,
then $f$ is nonnegative on its domain.

proof) If $n < 3$, then the statement holds trivially. Assume $n \geq 3$ and let $f$ be a function satisfying the hypotheses in the statement. For $2 \leq k \leq n - 1$, we have

$$f(k) + f(n - 1) + f(n - k + 1) \geq f(k - 1) + f(n - k) + f(n)$$

by letting $a = 1$, $b = k - 1$, and $c = n - k$. Therefore,

$$\sum_{k=2}^{n-1} (f(k) + f(n - 1) + f(n - k + 1)) \geq \sum_{k=2}^{n-1} (f(k - 1) + f(n - k) + f(n)),$$

which is simplified to

$$nf(n - 1) \geq (n - 2)f(n).$$

Since $f(n) \geq 0$, we have $f(n - 1) \geq 0$. Therefore, by induction on $n$, the statement holds for all positive integers.