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1 Solution

It suffices to prove the following statement, which obviously implies the statement in the problem:

For every positive integer n, if $f : \{1, 2, \ldots, n\} \to \mathbb{R}$ is a function such that (i) $f(1) = 0$, (ii) $f(n) \geq 0$, and (iii) $f(a+b) + f(b+c) + f(c+a) \geq f(a) + f(b) + f(c) + f(a+b+c)$ for all $1 \leq a, b, c \leq n-2$ with $a+b+c \leq n$, then f is nonnegative on its domain.

proof) If $n < 3$, then the statement holds trivially. Assume $n \geq 3$ and let f be a function satisfying the hypotheses in the statement. For $2 \leq k \leq n-1$, we have

$$f(k) + f(n-1) + f(n-k+1) \geq f(k-1) + f(n-k) + f(n)$$

by letting $a = 1$, $b = k-1$, and $c = n-k$. Therefore,

$$\sum_{k=2}^{n-1} \left(f(k) + f(n-1) + f(n-k+1) \right) \geq \sum_{k=2}^{n-1} \left(f(k-1) + f(n-k) + f(n) \right),$$

which is simplified to

$$nf(n-1) \geq (n-2)f(n).$$

Since $f(n) \geq 0$, we have $f(n-1) \geq 0$. Therefore, by induction on n, the statement holds for all positive integers.