Simultaneously diagonalizable matrices

KAIST POW 2023-22

김찬우, 연세대학교 수학과 22학번

25 November 2023

Problem. Does there exist a nontrivial subgroup G of $\mathbf{GL}(10, \mathbb{C})$ such that each element in G is diagonalizable but the set of all the elements of G is not simultaneously diagonalizable?

Solution. Let's lay out the facts we need.

Fact 1. Every element of a finite subgroup G of $\mathbf{GL}(10, \mathbb{C})$ is diagonalizable.

Proof. For any $A \in G$, $A^{|G|} = I$, so the minimal polynomial of A divides $x^{|G|} - 1 = 0$. Since $x^{|G|} - 1 = 0$ has $|G|$ distinct roots in $\mathbb{C}$, A is diagonalizable. ■

Fact 2. If A, $B \in \mathbf{GL}(10, \mathbb{C})$ are diagonalizable, then they are simultaneously diagonalizable only if $AB = BA$.

Proof. If A and B are simultaneously diagonalizable, there exists $P \in \mathbf{GL}(10, \mathbb{C})$ such that both PAP^{-1} and PBP^{-1} are diagonal matrices. Since two diagonal matrices commute, we can see that

$$AB = P^{-1}\left(PAP^{-1}\right)\left(PBP^{-1}\right)P = P^{-1}\left(PBP^{-1}\right)\left(PAP^{-1}\right)P = BA$$

■

Define A, $B \in \mathbf{GL}(10, \mathbb{C})$ as

$$A = \begin{pmatrix} \omega^0 & 0 & 0 & \cdots & 0 \\ 0 & \omega^1 & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^9 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $\omega = e^{\frac{2\pi i}{10}}$. Then we have $A^{10} = B^{10} = I$ and $AB = \omega^{-1}BA$. Thus the subgroup $G = \langle A, B \rangle$ of $\mathbf{GL}(10, \mathbb{C})$ generated by two elements A and B is finite, and is not Abelian since $AB = \omega^{-1}BA \neq BA$. Since $G \neq \mathbf{GL}(n, \mathbb{C})$ is obvious, the previous two facts show that G is the desired nontrivial subgroup of $\mathbf{GL}(10, \mathbb{C})$. □