# POW2023-21 A Limit Solution

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November 19, 2023

We have

$$\begin{aligned}\left(1 \pm \frac{1}{x}\right)^x &= e^{x \log\left(1 \pm \frac{1}{x}\right)} \\ &= e^{\pm 1 - \frac{1}{2x} + O\left(\frac{1}{x^2}\right)} \\ &= e^{\pm 1}\left(1 - \frac{1}{2x} + O\left(\frac{1}{x^2}\right)\right)\left(1 + O\left(\frac{1}{x^2}\right)\right) \\ &= e^{\pm 1} - \frac{e^{\pm 1}}{2x} + O\left(\frac{1}{x^2}\right)\end{aligned}$$

as $x \to \infty$. Also,

$$\begin{aligned}n^n + (n-1)^{n-1} \le \sum_{k=1}^{n} k^k &\le n^n + (n-1)^{n-1} + (n-2)^{n-2} + (n-3)^{n-3} + \cdots + 1^1 \\ &\le n^n + (n-1)^{n-1} + n^{n-2} + n \cdot n^{n-3}\end{aligned}$$

implies that

$$\begin{aligned}\sum_{k=1}^{n} k^k &= n^n \left(1 + \left(1 - \frac{1}{n}\right)^n \cdot \frac{1}{n-1} + O\left(\frac{1}{n^2}\right)\right) \\ &= n^n \left(1 + \left(\frac{1}{e} + O\left(\frac{1}{n}\right)\right)\left(\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) + O\left(\frac{1}{n^2}\right)\right) \\ &= n^n \left(1 + \frac{1}{en} + O\left(\frac{1}{n^2}\right)\right)\end{aligned}$$

as $n \to \infty$. Thus by dividing, the given sequence is

$$\begin{aligned}&(n+2)\left(1 + \frac{1}{n+1}\right)^{n+1}\left(1 + \frac{1}{e(n+2)} + O\left(\frac{1}{n^2}\right)\right)\left(1 - \frac{1}{e(n+1)} + O\left(\frac{1}{n^2}\right)\right) \\ &- (n+1)\left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{e(n+1)} + O\left(\frac{1}{n^2}\right)\right)\left(1 - \frac{1}{en} + O\left(\frac{1}{n^2}\right)\right)\end{aligned}$$

which is

$$(n+2)\left(e + \frac{e}{2(n+1)} + O\left(\frac{1}{n^2}\right)\right)\left(1 + O\left(\frac{1}{n^2}\right)\right) - (n+1)\left(e + \frac{e}{2n} + O\left(\frac{1}{n^2}\right)\right)\left(1 + O\left(\frac{1}{n^2}\right)\right)$$

and

$$(n+2)\left(e + \frac{e}{2n} + O\left(\frac{1}{n^2}\right)\right) - (n+1)\left(e + \frac{e}{2n} + O\left(\frac{1}{n^2}\right)\right) = e + O\left(\frac{1}{n}\right) \to e$$

as $n \to \infty$. Thus the limit is $e$.