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A sequence with small tail

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1 Problem

Can we find a sequence a_i, $i = 0, 1, 2, \ldots$ with the following property: for each given integer $n \geq 0$, we have

$$\lim_{L\to\infty} \sum_{i=0}^{L} 2^{ni}|a_i| \leq 23^{(n+11)^{10}} \quad \text{and} \quad \lim_{L\to\infty} \sum_{i=0}^{L} 2^{ni} a_i = (-1)^n?$$

2 Solution

Yes. For integers $k \geq 0$ and $n \geq k$, define

$$a_{k,n} = \left(\prod_{i=0}^{k-1} \frac{2^i + 1}{2^i - 2^k} \right) \cdot \left(\prod_{i=k+1}^{n} \frac{2^i + 1}{2^i - 2^k} \right).$$

At $n = k$, the latter factor is interpreted as 1, being an empty product. We claim that the limit $a_k = \lim_{n\to\infty} a_{k,n}$ exists for each k, and the sequence $\{a_k\}$ satisfies the given condition.

2.1 $a_{k,n}$ forms a solution to the linear system

Let us call the collection of equations $\lim_{L\to\infty} \sum_{i=0}^{L} 2^{ni} a_i = (-1)^n$ for all $n \geq 0$ by the *grand linear system* of countably many variables a_i, just for fun. Incidently, $a_{k,n}$ forms a solution to first few equations obtained from the grand linear system.

Proposition 2.1. *Let $L \geq 0$ be an integer. For $0 \leq n \leq L$, the equation*

$$\sum_{k=0}^{L} 2^{nk} a_{k,L} = (-1)^n$$

holds.

Proof. Omitted. □

2.2 a_k forms a solution to the grand linear system

Firstly, we claim that $a_{k,n}$ is convergent as $n \to \infty$ for every $k \geq 0$.

Lemma 2.2. *For $0 \leq x \leq 1$, $1 + x \leq e^x \leq 1 + 2x$ holds.*

Proof. Considering the series expansion $e^x = 1+x+\sum_{n=2}^{\infty} \frac{x^n}{n!}$, the first inequality is well-known. For the second inequality, it suffices to observe that e^x is convex, and that $e^1 \leq 1+2$ at $x = 1$. For $0 \leq x \leq 1$, x is expressed as $0 \cdot (1-x) + 1 \cdot x$, and we have $e^x \leq (1-x)e^0 + xe^1 = 1 + (e^1 - 1)x \leq 1 + 2x$. □

Proposition 2.3. *Let k, L, M be integers such that $M > L > k \geq 0$. Then the inequality*

$$\prod_{i=L+1}^{M} \frac{2^i + 1}{2^i - 2^k} \leq 1 + 8 \cdot \frac{2^k}{2^L}$$

holds.

Proof. Firstly, we have

$$\prod_{i=L+1}^{M} \frac{2^i + 1}{2^i - 2^k} \leq \exp\left(\sum_{i=L+1}^{M} \frac{2^k + 1}{2^i - 2^k}\right)$$

holds, as $\frac{2^i+1}{2^i-2^k} = 1 + \frac{2^k+1}{2^i-2^k} \leq \exp\left(\frac{2^k+1}{2^i-2^k}\right)$ for each i, by lemma 2.2.

Next, we have

$$\frac{1}{2^i - 2^k} = \frac{1}{2^i}\left(1 + \frac{2^k}{2^i - 2^k}\right) \leq \frac{1}{2^i}\left(1 + \frac{2^k}{2^{L+1} - 2^k}\right) = \frac{1}{2^i} \cdot \frac{2^{L+1}}{2^{L+1} - 2^k}$$

for $i \geq L + 1$, hence

$$\sum_{i=L+1}^{M} \frac{2^k + 1}{2^i - 2^k} \leq (2^k + 1) \cdot \frac{2^{L+1}}{2^{L+1} - 2^k} \sum_{i=L+1}^{M} \frac{1}{2^i}.$$

As $\frac{2^k+1}{2^k} \leq 2$, $\frac{2^{L+1}}{2^{L+1}-2^k} \leq 2$, and $\sum_{i=L+1}^{M} \frac{1}{2^i} \leq \sum_{i=L+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^L}$, we have an upper bound

$$\sum_{i=L+1}^{M} \frac{2^k + 1}{2^i - 2^k} \leq 4 \cdot \frac{2^k}{2^L}.$$

By lemma 2.2, we have

$$\exp\left(\sum_{i=L+1}^{M} \frac{2^k + 1}{2^i - 2^k}\right) \leq 1 + 8 \cdot \frac{2^k}{2^L},$$

and deduce the conclusion. □

Proposition 2.4. *For $k \geq 0$, the sequence $\{a_{k,n}\}$ is monotone and bounded.*

Proof. We have $a_{k,n+1} = \frac{2^{n+1}+1}{2^{n+1}-2^k} a_{k,n}$. As $\frac{2^{n+1}+1}{2^{n+1}-2^k} > 1$ for $n \geq k$, $|a_{k,n}|$ is increasing and $a_{k,n}$ is monotone.

Applying proposition 2.3 for $M \to \infty$, we have

$$\begin{aligned}
|a_{k,n}| &= \prod_{i=0}^{k-1} \frac{2^i+1}{2^k-2^i} \cdot \prod_{i=k+1}^{k+10} \frac{2^i+1}{2^k-2^i} \cdot \prod_{i=k+11}^{\infty} \frac{2^i+1}{2^i-2^k} \\
&\leq \prod_{i=0}^{k-1} \frac{2^i+1}{2^k-2^i} \cdot \prod_{i=k+1}^{k+10} \frac{2^i+1}{2^k-2^i} \cdot \left(1 + 8 \cdot \frac{2^k}{2^{k+10}}\right),
\end{aligned}$$

where the bound is a finite expression. □

Corollary 2.5. *For $k \geq 0$, the sequence $\{a_{k,n}\}$ converges.*

Denote this limit as $a_k = \lim_{n\to\infty} a_{k,n}$.

Proposition 2.6. *For $k \geq 1$, we have an inequality*

$$|a_{k,k}| \leq \frac{2^{k(k+1)/2}}{2^{(k-1)^2}} = \frac{1}{2} \cdot \frac{2^{5k/2}}{2^{k^2/2}}.$$

Proof. Note that $a_{k,k} = \prod_{i=0}^{k-1} \frac{2^i+1}{2^k-2^i}$. Observe that

$$\prod_{i=0}^{k-1} (2^k - 2^i) \geq \prod_{i=0}^{k-1} (2^k - 2^{k-1}) = 2^{(k-1)^2}$$

and

$$\prod_{i=0}^{k-1} (2^i + 1) \leq \prod_{i=0}^{k-1} (2^i + 2^i) = 2^{k(k+1)/2}.$$

We deduce the conclusion by combining two inequalities. □

Proposition 2.7. *Let k, L, M be integers such that $M > L$, $k \geq 0$, and L is sufficiently greater than k. Then we have an inequality*

$$|a_{k,M} - a_{k,L}| \leq C \cdot \frac{2^{7k/2}}{2^{k^2/2}} \cdot \frac{1}{2^L},$$

for some constant C, where $C = 4e^4$ is sufficient.

Proof. We have

$$\begin{aligned}
|a_{k,M} - a_{k,L}| &= \prod_{i=0}^{k-1} \frac{2^i+1}{2^k-2^i} \cdot \prod_{i=k+1}^{L} \frac{2^i+1}{2^i-2^k} \left(\prod_{i=L+1}^{M} \frac{2^i+1}{2^i-2^k} - 1 \right) \\
&\leq \frac{1}{2} \cdot \frac{2^{5k/2}}{2^{k^2/2}} \cdot \prod_{i=k+1}^{L} \frac{2^i+1}{2^i-2^k} \cdot 8 \cdot \frac{2^k}{2^L}.
\end{aligned}$$

As we might bound the product factor by a constant, we deduce the conclusion: a_k forms a solution to the grand linear system. □

Proposition 2.8. *Let $n \geq 0$ be an integer, $\varepsilon > 0$ be a positive real number. There exists an integer $L_0 = L(n, \varepsilon)$ such that, for every $L \geq L_0$ and $M > L$,*

$$\left| \sum_{k=0}^{L} 2^{nk} a_{k,M} - \sum_{k=0}^{L} 2^{nk} a_{k,L} \right| \leq \varepsilon.$$

Proof. We have

$$\left| \sum_{k=0}^{L} 2^{nk} a_{k,M} - \sum_{k=0}^{L} 2^{nk} a_{k,L} \right| \leq \sum_{k=0}^{L} 2^{nk} |a_{k,M} - a_{k,L}|.$$

If L is sufficiently greater than k and $2n + 7$, we have an upper bound

$$C \cdot \frac{1}{2^L} \sum_{k=0}^{L} \frac{2^{(n+7/2)k}}{2^{k^2/2}} = C \cdot \frac{1}{2^L} \sum_{k=0}^{2n+6} \frac{2^{(n+7/2)k}}{2^{k^2/2}} + C \cdot \frac{1}{2^L} \sum_{k=2n+7}^{L} \frac{2^{(n+7/2)k}}{2^{k^2/2}}.$$

Observe that the first sum is constant with respect to L, and the terms in the latter sum is at most 1, hence we have an upper bound of the form

$$\frac{D}{2^L} + \frac{C \cdot L}{2^L}.$$

This upper bound converges to zero as $L \to \infty$. □

Using the convergence of the finite sum $\sum_{k=0}^{L} 2^{nk} a_{k,M} \to \sum_{k=0}^{L} 2^{nk} a_k$ as $M \to \infty$, we have the following corollary.

Corollary 2.9. *Let $n \geq 0$ be an integer, $\varepsilon > 0$ be a positive real number. There exists an integer $L_0 = L(n, \varepsilon)$ such that, for every $L \geq L_0$,*

$$\left| \sum_{k=0}^{L} 2^{nk} a_k - \sum_{k=0}^{L} 2^{nk} a_{k,L} \right| \leq \varepsilon.$$

As $\sum_{k=0}^{L} 2^{nk} a_{k,L} = (-1)^n$, we deduce the conclusion.

Corollary 2.10. *For $n \geq 0$, $\lim_{L \to \infty} \sum_{k=0}^{L} 2^{nk} a_k = (-1)^n$.*

2.3 a_k satisfies the upper bound condition

With observed bounds for a_k, we can easily verify the sequence satisfying the given bound

$$\lim_{L \to \infty} \sum_{k=0}^{L} 2^{nk} |a_k| \leq 23^{(n+11)^{10}}$$

for every $n \geq 0$. Note that we have $|a_{k,k}| \leq \frac{2^{5k/2}}{2^{k^2/2}}$ for $k \geq 1$, and $\prod_{i=k+1}^{\infty} \frac{2^i+1}{2^i-2^k} \leq 2 \cdot \prod_{i=k+2}^{\infty} \frac{2^i+1}{2^i-2^k} \leq 2 \cdot (1+2) = 6$.