19 Let N be the number of ordered tuples of positive integers $(a_1, a_2, \cdots, a_{27})$ such that $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{27}} = 1$. Compute the remainder of N when N is divided by 3.

Solution. This lemma is useful.

Lemma. *For all pairs of integers $n \geq m \geq 1$,*

$$\frac{\gcd(n,m)}{n}\binom{n}{m} \in \mathbb{Z}.$$

Proof. Write $nx + my = \gcd(n,m)$ for $x, y \in \mathbb{Z}$. Then

$$\frac{\gcd(n,m)}{n}\binom{n}{m} = x\binom{n}{m} + y \cdot \frac{m}{n}\binom{n}{m} = x\binom{n}{m} + y\binom{n-1}{m-1}.$$

□

Define

$$S = \left\{ (a_1, a_2, \cdots, a_{27}) \in (\mathbb{Z}_+)^{27} \;\middle|\; \sum_{i=1}^{27} \frac{1}{a_i} = 1 \right\}.$$

Observe the following.

(i) $(27, \cdots, 27) \in S$.

(ii) For $x \neq y$ with $a > 0$, if $(\underbrace{x, \cdots, x}_{a}, \underbrace{y, \cdots, y}_{27-a})$ is in S, then so are its $\binom{27}{a}$ rearrangements.

(iii) For distinct x, y and z with $a > 0$ and $b > 0$, if $(\underbrace{x, \cdots, x}_{a}, \underbrace{y, \cdots, y}_{b}, \underbrace{z, \cdots, z}_{27-a-b})$ is in S, then so are its $\binom{27}{a}\binom{27-a}{b}$ rearrangements.

(iv) ... and so on.

Thus, $N = |S|$ must be $\left(\text{a linear combination of } \binom{27}{a} \text{ for each } a \text{ in } 1 \leq a \leq 26\right) + 1$. By Lemma, 3 divides $\binom{27}{a}$. Therefore, $N \equiv 1 \pmod 3$. □