Problem

Find all integers \( n \geq 8 \) such that there exists a simple graph with \( n \) vertices whose degrees are as follows:

1. \((n - 4)\) vertices of the graph are with degrees 4, 5, 6, \ldots, \( n - 2 \), \( n - 1 \), respectively.
2. The other 4 vertices are with degrees \( n - 2 \), \( n - 2 \), \( n - 1 \), \( n - 1 \), respectively.

Answer: 8, 9

Solution:

Note that degrees of vertices of the complement of such graph are 0, 1, 2, \ldots, \( n - 5 \), and 0, 0, 1, 1.

For \( n = 8, 9 \), the complement of desired graphs are as follows:

For \( n > 9 \), there is no such graph exists.

Suppose there exists such graph \( G \) with \( n \geq 9 \) vertices. Then \( \overline{G} \) has three isolated vertices. Let \( H \) be a subgraph of \( \overline{G} \) with \( m = n - 3 \) vertices, excluding isolated vertices. Note that degrees of vertices of \( H \) are 1, 1, 1, 2, 3, \ldots, \( m - 2 \) and \( m > 6 \). Let \( v_1, v_2, v_3 \) are vertices of \( H \) with degree 1 and \( w_1, w_2 \) are vertices of \( H \) with degree \( m - 3, m - 2 \), respectively.

Since \( w_2 \) is not adjacent to one vertices, at least two of \( v_1, v_2, v_3 \) are adjacent to \( w_2 \). Say \( v_1, v_2 \) are adjacent to \( w_2 \). Then, \( w_1 \) is not adjacent to \( v_1, v_2 \) and since \( w_1 \) is not adjacent to two vertices, \( w_1 \) is adjacent to \( v_3 \). In addition, \( w_1, w_2 \) are adjacent to all other vertices, and adjacent each other.

Let \( H' = H \setminus \{v_1, v_2, w_1, w_2, w_3\} \). Then, the number of vertices of \( H' \) is \( s = m - 5 = n - 8 > 1 \) and degrees of vertices are 0, 1, \ldots, \( m - 6 = s - 1 \), which is impossible since vertex of degree 0 is isolated, while vertex of degree \( s - 1 \) is adjacent to every other vertices.