Degrees of a graph

20238237 Lee Dohyeon POW 2023-18

Problem

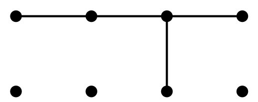
Find all integers $n \geq 8$ such that there exists a simple graph with n vertices whose degrees are as follows:

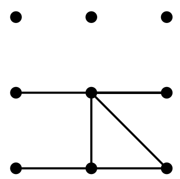
1. $(n-4)$ vertices of the graph are with degrees $4, 5, 6, \ldots, n-2, n-1$, respectively.
2. The other 4 vertices are with degrees $n-2, n-2, n-1, n-1$, respectively.

Answer: 8,9

Solution:

Note that degrees of vertices of the complement of such graph are $0, 1, 2, \ldots, n-5$, and $0, 0, 1, 1$. For $n = 8, 9$, the complement of desired graphs are as follows :





For $n > 9$, there is no such graph exists.

Suppose there exists such graph G with $n > 9$ vertices. Then $\overline{G}$ has three isolated vertices. Let H be a subgraph of $\overline{G}$ with $m = n-3$ vertices, excluding isolated vertices. Note that degrees of vertices of H are $1, 1, 1, 2, 3, \ldots, m-2$ and $m > 6$. Let v_1, v_2, v_3 are vertices of H with degree 1 and w_1, w_2 are vertices of H with degree $m-3, m-2$, respectively.

Since w_2 is not adjacent to one vertices, at least two of v_1, v_2, v_3 are adjacent to w_2. Say v_1, v_2 are adjacent to w_2. Then, w_1 is not adjacent to v_1, v_2 and since w_1 is not adjacent to two vertices, w_1 is adjacent to v_3. In addition, w_1, w_2 are adjacent to all other vertices, and adjacent each other.

Let $H' = H \backslash \{v_1, v_2, w_1, w_2, w_3\}$. Then, the number of vertices of H' is $s = m-5 = n-8 > 1$ and degrees of vertices are $0, 1, \ldots, m-6 = s-1$, which is impossible since vertex of degree 0 is isolated, while vertex of degree $s-1$ is adjacent to every other vertices.
