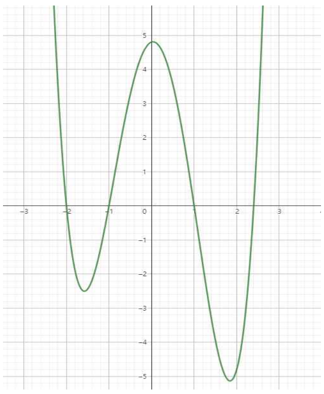
KAIST POW 2023-17

대전동산고등학교 2학년 서성욱

Let $f(x) = x^4 + (2-a)x^3 - (2a+1)x^2 + (a-2)x + 2a$ for some $a \geq 2$. Draw two tangent lines of its graph at the point $(-1,0)$ and $(1,0)$ and let P be the intersection point. Denote by T the area of the triangle whose vertices are $(-1,0)$, $(1,0)$ and P. Let A be the area of domain enclosed by the interval $[-1,1]$ and the graph of the function on this interval. Show that $T \leq 3A/2$.

Factorizing the functoin, we get

$$f = (x+2)(x+1)(x-1)(x-a)$$



<Figure 1.>

Figure 1. shows the graph of f. So

$$A = \int_{-1}^{1} f dx = \frac{8a}{3} - \frac{4}{15}.$$

At each point $(-1,0)$ and $(1,0)$, the slope is

$$\lim_{x \to -1} \frac{(x+2)(x+1)(x-1)(x-a)}{(x+1)} = 2(a+1),$$
$$\lim_{x \to 1} \frac{(x+2)(x+1)(x-1)(x-a)}{(x-1)} = 6(1-a)$$

so each tangent is $2(a+1)(x+1), 6(1-a)(x-1)$, and consequently, $x=\frac{3(a-1)}{2a-1}-1$, which is x coordinate of P. Thus, $P=(\frac{3(a-1)}{2a-1}-1, \frac{6(a^2-1)}{2a-1})$ and $T=\frac{6(a^2-1)}{2a-1}$.

So it can be easily shown that $T \le \frac{3A}{2}$ is equivalent with the inequality

$$5a^2-12a+16 \ge 0$$

in the region $a>\frac{1}{2}$ and since the equation has discriminant $D=36-80=-44<0$, the inequality holds true for all real number a, which of course includes all $a \ge 2$.