

Zeros in a sequence

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김찬우, 연세대학교 수학과 22학번

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Problem. Define the sequence x_n by $x_1 = 0$ and

$$x_n = x_{\lfloor n/2 \rfloor} + (-1)^{n(n+1)/2}$$

for $n \geq 2$. Find the number of $n \leq 2023$ such that $x_n = 0$.

Solution. Let $b_1 b_2 \dots b_k$ be a binary representation of the integer n with $b_1 = 1$. Then we can observe

- $b_{k-1} b_k = 00$ implies $n \equiv 0 \pmod{4}$, so $(-1)^{n(n+1)/2} = +1$
- $b_{k-1} b_k = 01$ implies $n \equiv 1 \pmod{4}$, so $(-1)^{n(n+1)/2} = -1$
- $b_{k-1} b_k = 10$ implies $n \equiv 2 \pmod{4}$, so $(-1)^{n(n+1)/2} = -1$
- $b_{k-1} b_k = 11$ implies $n \equiv 3 \pmod{4}$, so $(-1)^{n(n+1)/2} = +1$

Now define c_i as $+1$ if b_i and b_{i+1} are equal and -1 if they are different. It follows that

$$\begin{aligned} x_n &= x_{b_1 b_2 \dots b_k} \\ &= x_{b_1 b_2 \dots b_{k-1}} + c_{k-1} \\ &= x_{b_1 b_2 \dots b_{k-2}} + c_{k-2} + c_{k-1} \\ &\quad \vdots \\ &= \underbrace{x_{b_1}}_{=0} + c_1 + \dots + c_{k-2} + c_{k-1} \end{aligned}$$

Hence x_n can be completely determined by $(c_1, \dots, c_{k-1}) \in \{-1, +1\}^{k-1}$. Also, since $b_1 = 1$, there exists a one-to-one correspondence between (c_1, \dots, c_{k-1}) and $n = b_1 b_2 \dots b_k$. Now what we want to find is the number of elements in $\{-1, +1\}^{k-1}$ such that the sum of each component is zero. There is no such element when k is even, and if $k = 2m - 1$, then the number of such elements is

$$\binom{2m-2}{m-1}$$

since half of c_1, \dots, c_k must be 1 and other half -1 . Thus, the number of $n \leq 2047 = 2^{11} - 1$ such that $x_n = 0$ is

$$\binom{0}{0} + \binom{2}{1} + \binom{4}{2} + \binom{6}{3} + \binom{8}{4} + \binom{10}{5} = 351$$

The binary representation of 2023 is 11111100111, so we only miss the 11111101010 = 2026 when we restrict $n \leq 2023$, not $n \leq 2047$. Therefore the number of $n \leq 2023$ such that $x_n = 0$ is $351 - 1 = \boxed{350}$ \square

If we actually try to find a number that satisfies the condition, we get

1, 4, 6, 17, 19, 23, 25, 27, 29, 66, 68, 70, 72, 76, 78, 80, 88, 92, 94, 98, 100, 102, 104, 108, 110, 114, 116, 118, 122, 261, 265, 267, 269, 273, 275, 279, 281, 283, 285, 289, 291, 295, 303, 305, 307, 311, 313, 315, 317, 321, 323, 327, 335, 351, 353, 355, 359, 367, 369, 371, 375, 377, 379, 381, 389, 393, 395, 397, 401, 403, 407, 409, 411, 413, 417, 419, 423, 431, 433, 435, 439, 441, 443, 445, 453, 457, 459, 461, 465, 467, 471, 473, 475, 477, 485, 489, 491, 493, 501, 1034, 1042, 1044, 1046, 1050, 1058, 1060, 1062, 1064, 1068, 1070, 1074, 1076, 1078, 1082, 1090, 1092, 1094, 1096, 1100, 1102, 1104, 1112, 1116, 1118, 1122, 1124, 1126, 1128, 1132, 1134, 1138, 1140, 1142, 1146, 1154, 1156, 1158, 1160, 1164, 1166, 1168, 1176, 1180, 1182, 1184, 1200, 1208, 1212, 1214, 1218, 1220, 1222, 1224, 1228, 1230, 1232, 1240, 1244, 1246, 1250, 1252, 1254, 1256, 1260, 1262, 1266, 1268, 1270, 1274, 1282, 1284, 1286, 1288, 1292, 1294, 1296, 1304, 1308, 1310, 1312, 1328, 1336, 1340, 1342, 1344, 1376, 1392, 1400, 1404, 1406, 1410, 1412, 1414, 1416, 1420, 1422, 1424, 1432, 1436, 1438, 1440, 1456, 1464, 1468, 1470, 1474, 1476, 1478, 1480, 1484, 1486, 1488, 1496, 1500, 1502, 1506, 1508, 1510, 1512, 1516, 1518, 1522, 1524, 1526, 1530, 1546, 1554, 1556, 1558, 1562, 1570, 1572, 1574, 1576, 1580, 1582, 1586, 1588, 1590, 1594, 1602, 1604, 1606, 1608, 1612, 1614, 1616, 1624, 1628, 1630, 1634, 1636, 1638, 1640, 1644, 1646, 1650, 1652, 1654, 1658, 1666, 1668, 1670, 1672, 1676, 1678, 1680, 1688, 1692, 1694, 1696, 1712, 1720, 1724, 1726, 1730, 1732, 1734, 1736, 1740, 1742, 1744, 1752, 1756, 1758, 1762, 1764, 1766, 1768, 1772, 1774, 1778, 1780, 1782, 1786, 1802, 1810, 1812, 1814, 1818, 1826, 1828, 1830, 1832, 1836, 1838, 1842, 1844, 1846, 1850, 1858, 1860, 1862, 1864, 1868, 1870, 1872, 1880, 1884, 1886, 1890, 1892, 1894, 1896, 1900, 1902, 1906, 1908, 1910, 1914, 1930, 1938, 1940, 1942, 1946, 1954, 1956, 1958, 1960, 1964, 1966, 1970, 1972, 1974, 1978, 1994, 2002, 2004, 2006, 2010