POW 2023-15
An inequality for complex polynomials

2018 김기수

September 22, 2023

1 Problem

Let $p(z), q(z)$ and $r(z)$ be polynomials with complex coefficients in the complex plane. Suppose that $|p(z)| + |q(z)| \leq |r(z)|$ for every z. Show that there exist two complex numbers a, b such that $|a|^2 + |b|^2 = 1$ and $ap(z) + bq(z) = 0$ for every z.

2 Solution

Proposition 2.1. *Let $p(z), r(z)$ be polynomials with complex coefficients in the complex plane. Suppose that $|p(z)| \leq |r(z)|$ for every z except at 0. Then $|p(z)| \leq |r(z)|$ holds for every z.*

Proof. Since $|-| : \mathbb{C} \to \mathbb{R}$ and a polynomial function $\mathbb{C} \to \mathbb{C}$ are continuous, we have $|p(0)| = \left|\lim_{z\to 0} p(z)\right| = \lim_{z\to 0} |p(z)|$, and similarly $|r(0)| = \lim_{z\to 0} |r(z)|$. □

Lemma 2.2. *Let $p(z)$ be a polynomial with complex coefficients in the complex plane, c a constant complex number. Suppose that $|p(z)| \leq c$ for every z. Then $p(z)$ is a constant, i.e. there exists a complex number p_0 such that $p(z) = p_0$ for every z.*

Proof. While we can apply the Liouville's theorem, it suffices to observe that the magnitude of any nonzero nonconstant polynomial on the entire complex plane is unbounded. □

Lemma 2.3. *Let $p(z), r(z)$ be polynomials with complex coefficients in the complex plane. If $|p(z)| \leq |r(z)|$ for every z, then there exists a complex number c such that $p(z) = c \cdot r(z)$ for every z.*

Proof. Induction on the degree d of r. If $d = 0$, then it is the case of the lemma 2.2.

Suppose that the proposition holds for degree d, and assume the degree of r is $d + 1$. By the fundamental theorem of algebra, r has a zero. Say α be a zero of r, and write $r(z) = (z - \alpha)r^{(1)}(z)$ for a degree d polynomial $r^{(1)}(z)$. Let $\tilde{p}(z) = p(z + \alpha)$, $\tilde{r}^{(1)}(z) = r^{(1)}(z + \alpha)$, $\tilde{r}(z) = r(z + \alpha) = z \cdot \tilde{r}^{(1)}(z)$. We have $|\tilde{p}(z)| \leq |z| \cdot r^{(1)}(z + \alpha)$ for every z. At $z = 0$, we have $|\tilde{p}(0)| \leq 0$ hence $\tilde{p}(0) = 0$. Writing $\tilde{p}(z) = z \cdot \tilde{p}^{(1)}(z)$ for a polynomial $\tilde{p}^{(1)}(z)$, we have $|z| \cdot |\tilde{p}^{(1)}(z)| \leq |z| \cdot |\tilde{r}^{(1)}(z)|$ for every z, equivalently $|\tilde{p}^{(1)}(z)| \leq |\tilde{r}^{(1)}(z)|$ for every z except at 0. Applying proposition 2.1, we have $|\tilde{p}^{(1)}(z)| \leq |\tilde{r}^{(1)}(z)|$ for every z. Applying induction hypothesis for degree d polynomial $\tilde{r}^{(1)}(z)$, choose a constant c such that $\tilde{p}^{(1)}(z) = c \cdot \tilde{r}^{(1)}(z)$. Then we have $p^{(1)}(z+\alpha) = c \cdot r^{(1)}(z+\alpha)$ for every z, hence $p(z) = c \cdot r(z)$. This concludes the induction step. □

As the absolute values are nonnegative, we have $|p(z)| \leq |r(z)|$ and $|q(z)| \leq |r(z)|$ for every z. Using lemma 2, choose complex numbers u, v so that $p(z) = u \cdot r(z)$, $q(z) = v \cdot r(z)$. If one of u and v is zero, say $u = 0$ without loss of generality, then we can choose $a = 1$ and $b = 0$, which gives $|a|^2 + |b|^2 = 1$ and $ap(z) + bq(z) = 0$. Suppose both u and v are nonzero. Choose $a = \frac{v}{(|u|^2+|v|^2)^{1/2}}$, $b = \frac{-u}{(|u|^2+|v|^2)^{1/2}}$. Then we have $|a|^2 + |b|^2 = 1$, and $ap(z) + bq(z) = \frac{1}{(|u|^2+|v|^2)^{1/2}}(v \cdot ur(z) - u \cdot vr(z)) = 0$ for every z.