1 Problem

Let $p(z), q(z)$ and $r(z)$ be polynomials with complex coefficients in the complex plane. Suppose that $|p(z)| + |q(z)| \leq |r(z)|$ for every $z$. Show that there exist two complex numbers $a, b$ such that $|a|^2 + |b|^2 = 1$ and $ap(z) + bq(z) = 0$ for every $z$.

2 Solution

Proposition 2.1. Let $p(z), r(z)$ be polynomials with complex coefficients in the complex plane. Suppose that $|p(z)| \leq |r(z)|$ for every $z$ except at 0. Then $|p(z)| \leq |r(z)|$ holds for every $z$.

Proof. Since $|\cdot : \mathbb{C} \to \mathbb{R}$ and a polynomial function $\mathbb{C} \to \mathbb{C}$ are continuous, we have $|p(0)| = \lim_{z \to 0} |p(z)| = \lim_{z \to 0} |p(z)|$, and similarly $|r(0)| = \lim_{z \to 0} |r(z)|$. \quad \Box

Lemma 2.2. Let $p(z)$ be a polynomial with complex coefficients in the complex plane, $c$ a constant complex number. Suppose that $|p(z)| \leq c$ for every $z$. Then $p(z)$ is a constant, i.e. there exists a complex number $p_0$ such that $p(z) = p_0$ for every $z$.

Proof. While we can apply the Liouville’s theorem, it suffices to observe that the magnitude of any nonzero nonconstant polynomial on the entire complex plane is unbounded. \quad \Box

Lemma 2.3. Let $p(z), r(z)$ be polynomials with complex coefficients in the complex plane. If $|p(z)| \leq |r(z)|$ for every $z$, then there exists a complex number $c$ such that $p(z) = c \cdot r(z)$ for every $z$.

Proof. Induction on the degree $d$ of $r$. If $d = 0$, then it is the case of the lemma 2.2.

Suppose that the proposition holds for degree $d$, and assume the degree of $r$ is $d + 1$. By the fundamental theorem of algebra, $r$ has a zero. Say $a$ be
a zero of $r$, and write $r(z) = (z - \alpha)r^{(1)}(z)$ for a degree $d$ polynomial $r^{(1)}(z)$.

Let $\tilde{p}(z) = p(z + \alpha)$, $\tilde{r}^{(1)}(z) = r^{(1)}(z + \alpha)$, $\tilde{r}(z) = r(z + \alpha) = z \cdot r^{(1)}(z)$.

We have $|\tilde{p}(z)| \leq |z| \cdot r^{(1)}(z + \alpha)$ for every $z$. At $z = 0$, we have $|\tilde{p}(0)| \leq 0$ hence $\tilde{p}(0) = 0$. Writing $\tilde{p}(z) = p(z + \alpha)$ for a polynomial $\tilde{p}(z)$, we have $|z| \cdot |\tilde{p}(z)| \leq |z| \cdot |r^{(1)}(z)|$ for every $z$, equivalently $|\tilde{p}(z)| \leq |\tilde{r}^{(1)}(z)|$ for every $z$ except at 0. Applying proposition 2, we have $|\tilde{p}^{(1)}(z)| \leq |\tilde{r}^{(1)}(z)|$ for every $z$. Applying induction hypothesis for degree $d$ polynomial $\tilde{r}^{(1)}(z)$, choose a constant $c$ such that $\tilde{p}^{(1)}(z) = c \cdot \tilde{r}^{(1)}(z)$. Then we have $p^{(1)}(z + \alpha) = c \cdot r^{(1)}(z + \alpha)$ for every $z$, hence $p(z + \alpha) = c \cdot r(z)$. This concludes the induction step.

As the absolute values are nonnegative, we have $|p(z)| \leq |r(z)|$ and $|q(z)| \leq |r(z)|$ for every $z$. Using lemma 2, choose complex numbers $u, v$ so that $p(z) = u \cdot r(z)$, $q(z) = v \cdot r(z)$. If one of $u$ and $v$ is zero, say $u = 0$ without loss of generality, then we can choose $a = 1$ and $b = 0$, which gives $|a|^2 + |b|^2 = 1$ and $ap(z) + bq(z) = 0$. Suppose both $u$ and $v$ are nonzero. Choose $a = \frac{v}{(|u|^2 + |v|^2)^{1/2}}$, $b = \frac{-u}{(|u|^2 + |v|^2)^{1/2}}$. Then we have $|a|^2 + |b|^2 = 1$, and $ap(z) + bq(z) = \frac{1}{(|u|^2 + |v|^2)^{1/2}}(v \cdot ur(z) - u \cdot vr(z)) = 0$ for every $z$. 2