

POW 2023-14: Dividing polynomials

수리과학과 20학번 김준홍

Use $x^k - 1 = (x - 1)(x^{k-1} + \dots + x + 1)$ for $k \in \mathbb{N}$.

Then $\frac{f(t)}{g(t)} = \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)} = \frac{(t^{p(q-1)} + \dots + t^p + 1)}{(t^{q-1} + \dots + t + 1)}$. WLOG, assume that $p > q$.

For $1 \leq i \leq q - 1$, let $pi = a_i q + b_i$ for integer a, b , and $0 \leq b < q$.

As p, q are relatively prime, $\{b_1, \dots, b_{q-1}\} = \{1, \dots, q - 1\}$.

(It is trivial that $b_i \neq 0$. If $b_i = b_j$ satisfies for $i < j$, then $p(j-i) = (a_j - a_i)q$, so contradiction.)

Then $t^{p(q-1)} + \dots + t^p + 1 = (t^{p(q-1)} - t^{b_{q-1}}) + \dots + (t^p - t^{b_1}) + t^{q-1} + \dots + 1$.

Note that $t^{pi} - t^{bi} = t^{bi}(t^{a_i q} - 1) = t^{bi}(t^q - 1)(t^{q(a_i-1)} + \dots + t^q + 1) = t^{bi}(t - 1)(t^{q(a_i-1)} + \dots + t^q + 1)(t^{q-1} + \dots + 1) = \{(t^{q(a_i-1)+b_i+1} + \dots + t^{q+b_i+1} + t^{b_i+1}) - (t^{q(a_i-1)+b_i} + \dots + t^{q+b_i} + t^{b_i})\}(t^{q-1} + \dots + 1)$.

So $\frac{f(t)}{g(t)} = 1 + \sum_{i=1}^{q-1} \{(t^{q(a_i-1)+b_i+1} + \dots + t^{q+b_i+1} + t^{b_i+1}) - (t^{q(a_i-1)+b_i} + \dots + t^{q+b_i} + t^{b_i})\}$.

As $b_i \neq b_j$ if $i \neq j$, every term is not same with other term with same sign.

Therefore, coefficient can be only 1, 0, -1.