Solution to POW2023-13

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**Problem.** Prove or disprove the existence of a function $f : [0, 1] \to [0, 1]$ with the following property: for any interval $(a, b) \subseteq [0, 1]$ with $a < b$, $f((a, b)) = [0, 1]$.

**Solution.** Define an equivalence relation $\sim$ on $[0, 1]$ to be $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Consider the set of equivalent classes $[0, 1]/\sim$. Notice that $[0, 1]/\sim$ has the same cardinality with $[0, 1]$ since

$$
\text{card}([0, 1]) = \text{card}([0, 1]/\sim) \text{card}(\mathbb{Q}) = \max(\text{card}([0, 1]/\sim), \aleph_0) = \text{card}([0, 1]/\sim).
$$

Hence there is a bijection $\varphi : [0, 1]/\sim \to [0, 1]$. Now define a function $f : [0, 1] \to [0, 1]$ by $x \mapsto \varphi([x])$. It follows that for any $y \in [0, 1]$, $f^{-1}(y) = [x]$ for some $x \in [0, 1]$. Note that $[x] = \{x + r \in [0, 1] : r \in \mathbb{Q}\}$ is dense in $[0, 1]$. Thus any given interval $(a, b) \subseteq [0, 1]$ contains a point $z \in [x]$, and hence $f(z) = y$. Therefore $f((a, b)) = [0, 1]$. \qed