Solution to POW2023-13

KAIST 수리과학과
20220669 조현준

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Problem. Prove or disprove the existence of a function $f : [0,1] \to [0,1]$ with the following property: for any interval $(a,b) \subseteq [0,1]$ with $a < b$, $f((a,b)) = [0,1]$.

Solution. Define an equivalence relation $\sim$ on $[0,1]$ to be $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Consider the set of equivalent classes $[0,1]/{\sim}$. Notice that $[0,1]/{\sim}$ has the same cardinality with $[0,1]$ since

$$\operatorname{card}([0,1]) = \operatorname{card}([0,1]/{\sim})\operatorname{card}(\mathbb{Q}) = \max(\operatorname{card}([0,1]/{\sim}), \aleph_0) = \operatorname{card}([0,1]/{\sim}).$$

Hence there is a bijection $\varphi : [0,1]/{\sim} \to [0,1]$. Now define a function $f : [0,1] \to [0,1]$ by $x \mapsto \varphi([x])$. It follows that for any $y \in [0,1]$, $f^{-1}(y) = [x]$ for some $x \in [0,1]$. Note that $[x] = \{x + r \in [0,1] : r \in \mathbb{Q}\}$ is dense in $[0,1]$. Thus any given interval $(a,b) \subseteq [0,1]$ contains a point $z \in [x]$, and hence $f(z) = y$. Therefore $f((a,b)) = [0,1]$. $\square$