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Problem

Let S be a set of distinct 20 integers. A set T_A is defined as $T_A := \{s_1 + s_2 + s_3 \mid s_1, s_2, s_3 \in S\}$. What is the smallest possible cardinality of T_A?

Solution

I'll find it for arbitrary set S with n distinct integer.

Let $S = \{s_1, s_2, s_3, \cdots, s_n\}$ with $s_i < s_{i+1}$ for $i = 1, 2, \cdots, n-1$. T_A contains at least n distinct integers $3s_1 < 3s_2 < \cdots < 3s_n$ since $3s_i = s_i + s_i + s_i$ for all $i = 1, 2, \cdots, n$. Furthermore, T_A contains at least $2(n-1)$ elements more since for every $i = 1, 2, \cdots, n-1$, there are two distinct elements $s_i + s_i + s_{i+1}$ and $s_i + s_{i+1} + s_{i+1}$ in T_A between $3s_i$ and $3s_{i+1}$. Thus T_A contains at least $3n - 2$ elements.

Let $S = \{0, 1, \cdots, n-1\}$. Then clearly $T_A = \{0, 1, \cdots, 3n-3\}$ and $|T_A| = 3n - 2$. Thus minimal cardinality of T_A is $3n - 2$. For $n = 20$ case, it is 58.