POW2023-09

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May 2023

1 Solution

If $n = 1$, then obviously the equality holds. Assume that the equality holds when the number of positive real numbers is less than n. Then, for positive real numbers $d_1, d_2, \ldots, d_n$,

$$\begin{aligned}
&\sum_{\sigma\in\mathbb{S}_n} \frac{1}{d_{\sigma(1)}(d_{\sigma(1)}+d_{\sigma(2)})\cdots(d_{\sigma(1)}+\cdots+d_{\sigma(n)})} \\
&= \frac{1}{d_1+\cdots+d_n}\sum_{\sigma\in\mathbb{S}_n} \frac{d_1+\cdots+d_n}{d_{\sigma(1)}(d_{\sigma(1)}+d_{\sigma(2)})\cdots(d_{\sigma(1)}+\cdots+d_{\sigma(n)})} \\
&= \frac{1}{d_1+\cdots+d_n}\sum_{\sigma\in\mathbb{S}_n} \frac{d_{\sigma(1)}+\cdots+d_{\sigma(n)}}{d_{\sigma(1)}(d_{\sigma(1)}+d_{\sigma(2)})\cdots(d_{\sigma(1)}+\cdots+d_{\sigma(n)})} \\
&= \frac{1}{d_1+\cdots+d_n}\sum_{\sigma\in\mathbb{S}_n} \frac{1}{d_{\sigma(1)}(d_{\sigma(1)}+d_{\sigma(2)})\cdots(d_{\sigma(1)}+\cdots+d_{\sigma(n-1)})} \\
&= \frac{1}{d_1+\cdots+d_n}\Bigg(\sum_{\sigma\in\mathbb{S}_n,\sigma(n)=1} \frac{1}{d_{\sigma(1)}(d_{\sigma(1)}+d_{\sigma(2)})\cdots(d_{\sigma(1)}+\cdots+d_{\sigma(n-1)})} \\
&+ \sum_{\sigma\in\mathbb{S}_n,\sigma(n)=2} \frac{1}{d_{\sigma(1)}(d_{\sigma(1)}+d_{\sigma(2)})\cdots(d_{\sigma(1)}+\cdots+d_{\sigma(n-1)})} \\
&+ \cdots + \sum_{\sigma\in\mathbb{S}_n,\sigma(n)=n} \frac{1}{d_{\sigma(1)}(d_{\sigma(1)}+d_{\sigma(2)})\cdots(d_{\sigma(1)}+\cdots+d_{\sigma(n-1)})}\Bigg) \\
&= \frac{1}{d_1+\cdots+d_n}\left(\frac{1}{d_2d_3\cdots d_n}+\frac{1}{d_1d_3\cdots d_n}\cdots+\frac{1}{d_1d_2\cdots d_{n-1}}\right) \\
&= \frac{1}{d_1d_2\cdots d_n}
\end{aligned}$$

Therefore, by induction on n, the equality in the problem holds.