Groups with a perfect commutator subgroup KAIST POW 2023-08

김찬우, 연세대학교 수학과 22학번

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Problem. Find a pair of nonisomorphic nonabelian groups so that their abelianizations are isomorphic and their commutator subgroups are perfect.

Solution. Consider the quaternion group Q and the dihedral group D_8 (of order 8). Then $Q \ncong D_8$ since there are not the same number of elements of order 2. Also, both Q and D_8 are nonabelian groups. Now by simple calculation, we can get

$$[Q,Q] = \{-1,1\}$$
 and $[D_8,D_8] = \{e,r^2\}$

It follows that their commutator subgroups are obviously perfect. Also, again by simple calculation, we can obtain that

$$Q/[Q,Q] = \langle i[Q,Q], j[Q,Q] \rangle \cong K_4$$

 $D_8/[D_8,D_8] = \langle s[D_8,D_8], r[D_8,D_8] \rangle \cong K_4$

Hence their abelianizations are both isomorphic to the Klein 4-group, so isomorphic. \Box