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Proof. Notice that

$$e^{if(x)} = \frac{1}{if'(x)} \frac{d}{dx} e^{if(x)}$$

which means

$$\int_a^b e^{if(x)} dx = \int_a^b \frac{1}{if'(x)} \frac{d}{dx} e^{if(x)} dx$$

Using integration by parts, we note that $f'(x) \geq t > 0$ and $f''(x) \geq 0$,

$$\begin{aligned}
\left| \int_a^b e^{if(x)} \mathrm{d}x \right| = \left| \int_a^b \frac{d}{dx} e^{if(x)} \frac{1}{if'(x)} dx \right| &= \left| \frac{e^{if(b)}}{if'(b)} - \frac{e^{if(a)}}{if'(a)} + \int_a^b \frac{f''(x)}{if'(x)^2} e^{if(x)} \mathrm{d}x \right| \\
&\leq \frac{1}{f'(b)} + \frac{1}{f'(a)} + \int_a^b \frac{f''(x)}{f'(x)^2} \, \mathrm{d}x \\
&\leq \frac{1}{f'(b)} + \frac{1}{f'(a)} + \left[\frac{1}{f'(a)} - \frac{1}{f'(b)} \right] \\
&= \frac{2}{f'(a)} \leq \frac{2}{t}
\end{aligned}$$

since $\left|e^{iy}\right| = 1$ for any real y as well as $|i| = 1$. This proves the desired bound for the given oscillatory integral. ∎

Remark. This bound cannot be improved further by noting that the equality holds if f is linear with slope t and $t(b - a)$ is an odd multiple of π.