POW 2023-06 Golden ratio and a function

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Short Answer: $\text{frac}(\phi \cdot n) < 2 - \phi$

Let $g(n) = \lfloor \phi \cdot n \rfloor$.
Let $\text{frac}(x) = x - \lfloor x \rfloor$.
Note that $\phi = \frac{1+\sqrt{5}}{2}$ satisfies $\phi^2 - \phi - 1 = 0$.
Using the identity $\text{frac}(A + B) = \text{frac}(\text{frac}(A) + B)$, we can obtain the followings.

If $2-\phi \leq \text{frac}(\phi \cdot n) < 1$, then we have $\text{frac}(\phi \cdot (n+1)) = \text{frac}(\phi \cdot n)+\phi-2$, $0 \leq \text{frac}(\phi \cdot (n+1)) < \phi - 1$, and $\lfloor \phi \cdot (n+1) \rfloor = \lfloor \phi \cdot n \rfloor + 2$.

If $0 \leq \text{frac}(\phi \cdot n) < 2-\phi$, then we have $\text{frac}(\phi \cdot (n+1)) = \text{frac}(\phi \cdot n)+\phi-1$, $\phi - 1 \leq \text{frac}(\phi \cdot (n+1)) < 1$, and $\lfloor \phi \cdot (n+1) \rfloor = \lfloor \phi \cdot n \rfloor + 1$.

Therefore, for all $n \in \mathbb{Z}_{>0}$,

$$g(n+1) = \begin{cases} g(n) + 2 & \text{if frac}(\phi \cdot n) \geq 2 - \phi \\ g(n) + 1 & \text{otherwise} \end{cases}$$

For all $n \in \mathbb{Z}_{>0}$,

$$\begin{aligned}
& g(g(n) - n + 1) = n \\
\iff\ & \lfloor \phi \cdot (g(n) - n + 1) \rfloor = n \\
\iff\ & n \leq \phi \cdot (g(n) - n + 1) < n + 1 \\
\iff\ & (\phi - 1) \cdot n \leq g(n) - n + 1 < (\phi - 1) \cdot (n + 1) && (\phi - 1 = \tfrac{1}{\phi}) \\
\iff\ & \phi \cdot n - 1 \leq g(n) < \phi \cdot (n + 1) - 2 \\
\iff\ & \phi \cdot n - 1 \leq \lfloor n \cdot \phi \rfloor < \phi \cdot (n + 1) - 2 \\
\iff\ & \lfloor \phi \cdot n \rfloor < \phi \cdot (n + 1) - 2 && (x - 1 \leq \lfloor x \rfloor \text{ for all } x \in \mathbb{R}) \\
\iff\ & \phi \cdot (n + 1) > g(n) + 2 \\
\iff\ & \phi \cdot (n + 1) \geq g(n) + 2 && (\phi \cdot (n+1) \notin \mathbb{Z} \text{ since } \phi \notin \mathbb{Q}) \\
\iff\ & \lfloor \phi \cdot (n + 1) \rfloor \geq g(n) + 2 \\
\iff\ & g(n + 1) \geq g(n) + 2 \\
\iff\ & \text{frac}(\phi \cdot n) \geq 2 - \phi && (\text{from the above equation})
\end{aligned}$$

So we obtain for all $n \in \mathbb{Z}_{>0}$,

$$f(f(n) - n + 1) \neq n \iff \text{frac}(\phi \cdot n) < 2 - \phi$$

which is the answer to the second part of the problem.

Also, we can write for all $n \in \mathbb{Z}_{>0}$,

$$g(n+1) = \begin{cases} g(n) + 2 & \text{if } g(g(n) - n + 1) = n \\ g(n) + 1 & \text{otherwise} \end{cases}$$

and $g(1) = \lfloor \phi \rfloor = 1$.

Now we want to show that for all $n \in \mathbb{Z}_{>0}$, $f(n) = g(n)$ using the induction.

$f(1) = 1 = g(1)$.

If $n \in \mathbb{Z}_{>0}$ and $f(k) = g(k)$ for all $k \in \mathbb{Z}_{>0}$ with $k \leq n$, then

$$\begin{aligned} g(n+1) &= \begin{cases} g(n) + 2 & \text{if } g(g(n) - n + 1) = n \\ g(n) + 1 & \text{otherwise} \end{cases} \\ &= \begin{cases} g(n) + 2 & \text{if } f(g(n) - n + 1) = n \\ g(n) + 1 & \text{otherwise} \end{cases} && \text{(induction hypothesis)} \\ &= \begin{cases} f(n) + 2 & \text{if } f(f(n) - n + 1) = n \\ f(n) + 1 & \text{otherwise} \end{cases} && \text{(induction hypothesis again)} \\ &= f(n+1) && \text{(definition of } f\text{)} \end{aligned}$$

Note that the induction hypothesis can be used in the second equality since

$$g(n) - n + 1 = \lfloor \phi \cdot n \rfloor - n + 1 \in \mathbb{Z}$$

$$g(n) - n + 1 = \lfloor \phi \cdot n \rfloor - n + 1 \leq \phi \cdot n - n + 1 = (\phi - 1) \cdot n + 1 < n + 1$$
$$g(n) - n + 1 = \lfloor \phi \cdot n \rfloor - n + 1 > \phi \cdot n - n = (\phi - 1) \cdot n > 0$$

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By the strong induction, $f(n) = g(n) = \lfloor \phi \cdot n \rfloor$ for all $n \in \mathbb{Z}_{>0}$.

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