POW 2023-05: Shuffle, multiply, and add

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Lemma) For $0 \le a_1 \le \cdots \le a_{kn}, \{a_1, \cdots, a_{kn}\} = \{b_1, \cdots, b_{kn}\}, \sum_{i=0}^{n-1}(\prod_{j=1}^{k} a_{ki+j}) \ge \sum_{i=0}^{n-1}(\prod_{j=1}^{k} b_{ki+j})$.

Proof) Use induction on n. $n = 1$ case is trivial, so consider $n = 2$. Let $A = \{a_1, \cdots, a_n\}$, $B = \{a_{n+1}, \cdots, a_{2n}\}, C = \{b_1, \cdots, b_n\}, D = \{b_{n+1}, \cdots, b_{2n}\}$. Note that $|D \cap B| + |C \cap B| = |B| = |C| = |C \cap A| + |C \cap B|$ and $|D \cap A| + |D \cap B| = |D| = |A| = |D \cap A| + |C \cap A|$, so $|D \cap B| = |C \cap A|, |D \cap A| = |C \cap B|$. $\Rightarrow \prod_{a \in D \cap B} a - \prod_{a \in C \cap A} a \ge 0$ and $\prod_{a \in C \cap B} a - \prod_{a \in D \cap A} a \ge 0$.

So $\sum_{i=0}^{1}(\prod_{j=1}^{k} a_{ki+j} - \prod_{j=1}^{k} b_{ki+j}) = (\prod_{a \in D \cap B} a - \prod_{a \in C \cap A} a)(\prod_{a \in C \cap B} a - \prod_{a \in D \cap A} a) \ge 0$, and statement holds.

Now assume statement holds for n. Apply lemma to $b_1, \cdots, b_{kn}$ and rearrange subsequence by ascending order. Next, apply lemma to $b_{k(n-1)+1}, \cdots, b_{k(n+1)}$ and rearrange subsequence by ascending order. Finally, apply lemma to $b_1, \cdots, b_{kn}$ again and rearrange subsequence by ascending order. Then rearranged sequence is same with $\{a_i\}$, so statement holds also in $n+1$. ■

Corollary) For $0 \le a_1 \le \cdots \le a_{kn}, \{a_1, \cdots, a_{kn}\} = \{b_1, \cdots, b_{kn}\},\ 0 \le c_1 \le \cdots \le c_n$, $\sum_{i=0}^{n-1} c_{i+1}(\prod_{j=1}^{k} a_{ki+j}) \ge \sum_{i=0}^{n-1} c_{i+1}(\prod_{j=1}^{k} b_{ki+j})$.

Proof) $\sum_{i=0}^{n-1} c_{i+1}(\prod_{j=1}^{k} b_{ki+j}) = c_1 \sum_{i=0}^{n-1}(\prod_{j=1}^{k} b_{ki+j}) + (c_2 - c_1)\sum_{i=1}^{n-1}(\prod_{j=1}^{k} b_{ki+j}) + \cdots + (c_n - c_{n-1})\prod_{j=1}^{k} b_{(n-1)i+j}$
$\le c_1 \sum_{i=0}^{n-1}(\prod_{j=1}^{k} a_{ki+j}) + (c_2 - c_1)\sum_{i=1}^{n-1}(\prod_{j=1}^{k} a_{ki+j}) + \cdots + (c_n - c_{n-1})\prod_{j=1}^{k} a_{(n-1)i+j} = \sum_{i=0}^{n-1} c_{i+1}(\prod_{j=1}^{k} a_{ki+j})$
by using lemma and $c_m - c_{m-1} \ge 0$. ■

Let $+$ denote 0 and positive integers, and $-$ denote negative integers. Then we can express $x_{3i+1}x_{3i+2}x_{3i+3}$ with one of $+++, ++-, +--, ---$(by rearranging the order of 3 numbers), and $(x_1, \cdots, x_{21})$ by $\{s_1, \cdots, s_7\}$, when $s_i \in \{+++, ++-, +--, ---\}$.

If $++-$ exists in $\{s_i\}$, then there should be $---$ or another $++-$ since there are total 10 $-$s. We can substitute $++-, ---$ to $+--, +--$ and $++-, ++-$ to $+++, +--$ so that 2 terms are changed from negative to positive. So only $+++, +--, ---$ should be used to get maximum.

If $---$ exists in $\{s_i\}$, then $\{s_i\} = \{+++,+++,+++,+--,+--,---,---\}$. Use lemma in $\{+++,+++,+++\}$, then smallest term has all 3 numbers among 0 to 4, which means its value is ≤ 24. Since at least one of $---$ has all numbers ≤ -2 so that its value is ≤ -24, $\{+++,---,---\}$ is negative. Change it to $\{+--,+--,+--\}$, which is positive, then result gets larger.

Therefore, only $+++,+--$ should be used, and it gives unique form $\{+++,+++,+--,+--,+--,+--,+--\}$. Use corollary to $\{+--,+--,+--,+--,+--\}$ with $\{b_i\} = \{-1,\cdots,-10\}, \{c_i\} = \{+,+,+,+,+\}$, then $-(2i-1), -2i$ should be in same term for $i = 1,2,3,4,5$.($-$ sign does not matter since they are cancelled)

If $x \neq 0$ for term $x(-1)(-2)$, use corollary for $a_i : 0 \leq (-1)(-2) \leq yz, c_i : 0 \leq x$ and change $0yz + x(-1)(-2)$ to $0(-1)(-2) + xyz$.

If $x \neq 10$ for term $x(-9)(-10)$, use corollary for $a_i : 0 \leq yz \leq (-9)(-10), c_i : x \leq 10$ and change $x(-9)(-10) + 10yz$ to $xyz + 10(-9)(-10)$.

If $x \neq 9$ for term $x(-7)(-8)$, use corollary for $a_i : 0 \leq yz \leq (-7)(-8), c_i : x \leq 9$ and change $x(-7)(-8) + 9yz$ to $xyz + 9(-7)(-8)$.

Now the form is $\{+++,+++,0(-1)(-2),+(-3)(-4),+(-5)(-6),9(-7)(-8),10(-9)(-10)\}$. Use lemma to $\{+++,+++\}$, and let $m(i) = (ith$ largest number in larger $+++$ term$)$. Then $m(1) \geq 6, m(2) \geq 5$. Use corollary to $\{+++,+(-3)(-4),+(-5)(-6)\}$ with $a_i : 0 \leq (-3)(-4) \leq (-5)(-6) \leq m(1)m(2), c_i : \{+,+,+\}$, then $m(1) = 8, m(2) \geq 6$ now. Use corollary to $\{++8,+(-3)(-4),+(-5)(-6)\}$ with $a_i : 0 \leq (-3)(-4) \leq (-5)(-6) \leq m(1)m(2), c_i : \{+,+,+\}$, then $+++ = 6\cdot 7\cdot 8$.

Now the form is $\{+++,6\cdot 7\cdot 8,0(-1)(-2),+(-3)(-4),+(-5)(-6),9(-7)(-8),10(-9)(-10)\}$. Let $p(i) = (ith$ largest number in $+++$ term$)$. Then $p(2) \leq 4, p(3) \leq 3$. Use corollary to $\{+++,+(-3)(-4),+(-5)(-6)\}$ with $a_i : 0 \leq p(3)p(2) \leq (-3)(-4) \leq (-5)(-6), c_i : \{+,+,+\}$, then $5(-5)(-6)$ is determined and $p(2), p(3) \neq 4$. Use corollary to $\{+++,+(-3)(-4)\}$ with $a_i : 0 \leq p(3)p(2) \leq (-3)(-4), c_i : \{+,+\}$, then $4(-3)(-4)$ is determined and $+++ = 1\cdot 2\cdot 3$.

$\therefore \{1\cdot 2\cdot 3, 6\cdot 7\cdot 8, 0(-1)(-2), 4(-3)(-4), 5(-5)(-6), 9(-7)(-8), 10(-9)(-10)\}$ gives greater or equal result than every arrangement, so maximum is $1\times 2\times 3 + 6\times 7\times 8 + 0\times(-1)\times(-2) + 4\times(-3)\times(-4) + 5\times(-5)\times(-6) + 9\times(-7)\times(-8) + 10\times(-9)\times(-10) = 1944$. $\blacksquare$