POW 2023-04

2021____ Chae Jiseok

March 30, 2023

Simple algebra shows that

$$
\begin{aligned}
(n^4+n^3+n^2+n+1) - \left(n^2+\frac{n}{2}\right)^2 &= \frac{3}{4}\left(n+\frac{2}{3}\right)^2 + \frac{2}{3}, \\
\left(n^2+\frac{n}{2}+\frac{1}{2}\right)^2 - (n^4+n^3+n^2+n+1) &= \frac{1}{4}(n-3)(n+1)
\end{aligned}
$$

and thus

$$\left(n^2+\frac{n}{2}\right)^2 < n^4+n^3+n^2+n+1 < \left(n^2+\frac{n}{2}+\frac{1}{2}\right)^2 \qquad \text{whenever } n>3 \text{ or } n<-1.$$

In other words, whenever $n > 3$ or $n < -1$, the quantity $\sqrt{n^4+n^3+n^2+n+1}$ lies in between two consecutive numbers in $\frac{1}{2}\mathbb{Z}$, so $n^4+n^3+n^2+n+1$ can never be a perfect square.

It remains to consider the cases when $-1 \leqslant n \leqslant 3$. Brute forcing gives us

$$
\begin{aligned}
(-1)^4+(-1)^3+(-1)^2+(-1)+1 &= 1, \\
0^4+0^3+0^2+0+1 &= 1, \\
1^4+1^3+1^2+1+1 &= 5, \\
2^4+2^3+2^2+2+1 &= 31, \\
3^4+3^3+3^2+3+1 &= 121.
\end{aligned}
$$

Therefore, the only integers n such that $n^4+n^3+n^2+n+1$ is a perfect square are -1, 0, and 3.