POW #2023-03

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Problem. Determine the minimum number of hyperplanes in $\mathbb{R}^n$ that do not contain the origin but they together cover all points in $\{0,1\}^n$ except the origin.

Solution. We claim that the answer is $n+1$. To show this, we begin with a well-known lemma. We defer its proof.

Lemma 1. *(Combinatorial Nullstellensatz) Let $\mathbb{F}$ be a field and $f \in \mathbb{F}[x_1, x_2, \cdots, x_n]$ be a polynomial. Say* $\deg(f) = d = \sum_{i=1}^n d_i$ *and assume that the coefficient of* $\prod_{i=1}^n x_i^{d_i} \neq 0$. *If $L_1, L_2, \cdots, L_n \subseteq \mathbb{F}$ satisfies $|L_i| > d_i$,*

$$\exists a_1 \in L_1, \cdots, a_n \in L_n \ s.t.\ f(a_1, a_2, \cdots, a_n) \neq 0.$$

First, we will show that we need at least n hyperplanes. Let $\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_m$ be hyperplanes covering $\{0,1\}^n$ except the origin. Say $\mathcal{H}_i = \{x \in \mathbb{R}^n : a_i \cdot x = b_i\}$ where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$. Define the polynomial P of degree m as

$$P(x_1, \cdots, x_n) = \prod_{i=1}^m (a_i \cdot x - b_i).$$

Now, we claim that $m = \deg(P) \geq n$.

Suppose to the contrary so that $m = \deg(P) < n$. We have $P(\vec{0}) \neq 0$ but $P(\vec{x}) = 0$ for all $\vec{x} \in \{0,1\}^n \backslash \{\vec{0}\}$. Define $Q(x_1, x_2, \cdots, x_n) = P(x_1, x_2, \cdots, x_n) - \alpha \prod_{i=1}^n (x_i - 1)$ by choosing appropriate nonzero α so that $Q(\vec{0}) = 0$. Note that $Q(\vec{x}) = 0$ for each $x \in \{0,1\}^n \setminus \{\vec{0}\}$, too (because both terms are zero). Note that the coefficient of $\prod_{i=1}^n x_i$ in Q is $\alpha \neq 0$. By Combinatorial Nullstellensatz, there exists $\vec{x} \in \{0,1\}^n$ such that $Q(x_1, x_2, \cdots, x_n) \neq 0$, which is a contradiction. Therefore, we have $m \geq n$.

Now, we will show that at most n hyperplanes suffice. There's straight forward construction:

- $H_i = \{(x_1, \cdots, x_n) : x_i = 1\}$ for each $1 \leq i \leq n$.

□

Proof of Lemma 1. We proceed with an induction on n. Note that the statement is trivial when $n = 1$ so we may assume that $n > 1$.

Without loss of generality, we may assume that $|L_n| = d_n + 1$. Now, consider a polynomial $f_n(x_n) = \prod_{t \in L_n} (x_n - t)$ having degree of $d_n + 1$. If we define a polynomial $h_n(x_n) = x_n^{d_n+1} - f_n(x_n)$, then the degree of $h_n(x_n)$ does not exceed d_n.

Now, consider a new polynomial $\tilde{f}$ obtained from f by repetitively replacing $x_n^{d_n+1}$ to $h_n(x_n)$. For $x_n \in L_n$, we have $x_n^{d_n+1} = h_n(x)$ so that $f(x_1, x_2, \cdots, x_n) = \tilde{f}(x_1, x_2, \cdots, x_n)$. We also know that the degree of $x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$ is same in both f and $\tilde{f}$.

Write $\tilde{f} = \sum_{i=0}^{d_n} g_i(x_1, x_2, \cdots, x_{n-1}) x_n^i$ and apply the induction hypothesis to g_{d_n}. Then, we have $a_1 \in L_2, \cdots, a_{n-1} \in L_{n-1}$ such that $g_{d_n}(a_1, \cdots, a_{n-1}) \neq 0$. After fixing $x_1 = a_1, \cdots, x_{n-1} = a_{n-1}$, $\tilde{f}$ becomes a polynomial with degree d_n so there exists $a_n \in L_n$ such that $\tilde{f}(a_1, a_2, \cdots, a_n) \neq 0$. □